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Stability and compensations of systems with multiple nonlinearities

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Stability and compensation of systems with multiple nonlinearities

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CHAPTER 1. INTRODUCTION

Stability, Instability and Boundedness

The concept of stability originated in mechanics where the position of rest of a rigid body is considered stable if the body returns to its original position after a small disturbance. This is a qualitative concept and even to-day the central problem of stability theory is to ascertain qualitative features of system behavior in the absence of knowledge of specific system solutions. The meaning of stability is much broader now and encompasses a broad range of system behavior. In fact, the designer and the analyst now have ways to study the qualitative behavior of a system to determine if such behavior is acceptable.

The main theoretical development in stability theory occurred at the end of the last century in the works of the Russian scientist Lyapunov. The Direct (or Second) Method of Lyapunov has been applied to the study of stability of certain systems by other Russian authors but for many years work in this area was confined to Russia. It was not until after World War II that these ideas spread to the West and in the last two decades the work done in applying and ad-vancing the Lyapunov approach to stability theory around the world has been extensive as the still growing literature in this field would suggest.

In the Direct Method of Lyapunov, a scalar function (known as the Lyapunov function) of the system state variables has to be found. If this function satisfies certain conditions, the system would behave in a particular manner. Hence, to show that a system is stable, a Lyapunov function with a particular set of properties has to be found. A system is considered stable, in the sense of Lyapunov, if the system state variables do not move away from the equilibrium state of the system when it is disturbed. The method can also determine asymptotic behavior of the system state trajectories or a limit to the disturbances the system can withstand. A still different set of properties of the Lyapunov function can establish instability of the system which means that the system trajectories move away from the equilibrium after a small perturbation. Sometimes a system may have an unstable equilibrium but its state trajectories may remain within certain bounds. This can also be determined by Lyapunov's method and such systems are known as bounded systems. This behavior is, in fact, a kind of stability and such systems are sometimes called Lagrange stable.

The Direct Method of Lyapunov was first applied to systems described by ordinary differential equations. Since then, it has been applied to systems represented by difference equations, partial differential equations, integral equations, functional equations, stochastic equations and combinations of these various types. The expansion of the theory itself and its successful application to a large class of systems have made Lyapunov's method a very powerful tool in the qualitative study of system behavior.

Large Interconnected Systems

Although the Direct Method of Lyapunov has been used extensively because of its versatility, there are some drawbacks in its application. Finding a Lyapunov function to establish the qualitative behavior of a system may not be easy since there is no general method known for finding such a function. Also, in general, the conditions for a particular type of qualitative behavior are sufficient and not necessary and so the choice of parameters of a system dictated by a particular Lyapunov function tend to be conservative. This means that even if a suitable function is found, there is no way of knowing how close this function is to the "best" function.

These difficulties get more acute as the systems get larger and more complex. For systems with a small number of variables, usually a "good" Lyapunov function can be found through a combination of intuition and trial and error. However, this becomes very difficult for large systems with many nonlinear elements. For this reason, it is sometimes advantageous to look at large systems as being composed of several small subsystems that are interconnected together in an appropriate manner. The qualitative behavior of such systems can often then be studied in terms of the simpler subsystems and the interconnecting structure. The problems of applying the Direct Method to large, high-order systems may be circumvented in this way.

Using a vector Lyapunov function, Bailey [5] established the first theorems in system stability in terms of its subsystems and linear interconnections. Extending this work, Piontkovskii and Rutkovskaya [19] analyzed the stability of two classes of composite systems with nonlinear, time-invariant interconnections. Thompson [22] used a scalar Lyapunov function to establish conditions for exponential stability. Araki, Ando and Kondo [3] used a particular Lyapunov function to obtain stability results for a class of systems described by difference equations. Michel and Porter [13] generalized this approach to systems with nonlinear, time-varying interconnections described by ordinary differential equations or difference equations.

This approach has been expanded to other aspects of stability analysis and has been applied to other types of systems. Grujić and Siljak [6], [7] using the vector Lyapunov function, obtained results for asymptotic stability and instability for systems having stable as well as unstable subsystems. Porter and Michel [20] studied bounded input bounded output stability using functional analysis. Michel [14], [15], and [16], has obtained results for asymptotic stability and has analyzed several classes of systems including stochastic systems and systems described by functional differential equations.

In this dissertation, the stability of a class of systems is analyzed using this approach and utilizing the frequency-domain Popov criterion. Also, new theorems are stated and proved for boundedness and instability of composite systems using a Lyapunov function that is a weighted sum of the Lyapunov functions of the subsystems. Finally, the conditions for the stability of a composite system that is made up of some stable and some unstable subsystems are established. This theorem suggests a method of compensation to make composite systems stable. Applications for all the results are provided by utilizing a variety of systems.

The Popov Criterion

A little over a decade ago, the Rumanian scientist Popov established a frequency domain criterion for the stability of a class of systems known as regulator systems characterized by a linear transfer function and a nonlinear element in cascade. Although Luré had already discovered a type of

Lyapunov function to analyze the stability of such systems, the Popov criterion had the obvious advantage of having a simple geometric interpretation. The Popov criterion for stability was a geometric criterion in the frequency domain whereas the function of Luré had to satisfy the algebraic conditions of Lyapunov in the time domain. Yakubovich [24] and Kalman [11] soon showed that the Popov criterion and the Lyapunov function of Luré were equivalent and outlined a method to construct the Lyapunov function from the geometry of the Popov criterion.

Several authors [2], [9], [10], [17] and [18], then tried to extend this work to systems with multiple nonlinearities and established what may be called the multi-dimensional Popov criterion. However, this lacks the simple geometric interpretation and depends on finding a matrix that will satisfy certain conditions. The method has some of the same disadvantages as that of finding a Lyapunov function.

In this dissertation, a class of systems with multiple nonlinearities, that can be decomposed into subsystems having a single nonlinear element in tandem with a linear transfer function, is analyzed using the simple Popov criterion. Using the approach for interconnected systems, the geometric frequency-domain criterion is used to determine the stability of the subsystems. The Kalman-Yakubovich lemma provides the Lyapunov function for the subsystems and from these functions and the nature of the interconnections, the stability of the composite system is determined in the usual manner. The advantage of using the simpler geometric formulation is obvious and in many cases, depending on the structure of the system, the results obtained should be better than from the previous methods.

Compensation

The main advantage in the interconnected systems approach is that the final result depends on a symmetric square matrix whose dimensions are the same as the number of subsystems. In other approaches the methods used are equivalent to dealing with a matrix of the same order as the systems themselves. However, it should be noted here that certain "measures" of the subsystems and interconnections are represented by scalars to form the smaller matrix and a degree of conservatism in the results has to be expected, depending on the structure of the system. Of course, this is the main disadvantage of this approach.

Usually the ultimate result (stability, instability or boundedness) depends on the definiteness of this matrix. Since the condition is sufficient, no conclusion can be reached when the condition is not satisfied. However, the structure of the matrix itself may suggest changes in the matrix and corresponding changes in the subsystems that will satisfy the stability condition. In this dissertation, a method is developed by which a negative feedback may be added to certain subsystems to make the composite system stable.

Outline

In Chapter 2 the necessary nomenclature used in the rest of the dissertation is established. Some prior results in the theory of interconnected systems are discussed in Chapter 3. In Chapter 4 the method of analysis for a class of systems using the Popov criterion and the interconnected systems approach is presented. The method is compared to other available methods and some examples are provided. In Chapter 5 some theorems on instability and boundedness of interconnected systems are stated, proved and illustrated with examples. The method for compensation is developed in Chapter 6 and applied in two examples. The conclusion and suggestions for further work are provided by Chapter 7. A detailed proof of the Kalman-Yakubovich Lemma and the equivalence of the Popov criterion and the Lyapunov function of Luré for regulator systems are included in the Appendix. The algorithm to construct the Lyapunov function from the Popov condition is clearly outlined.

CHAPTER 2. NOTATION

The symbol j is used for $\sqrt{-1}$, s is used for the Laplace transform operator and ω is used for frequency. The modulus of a complex or a real number a is denoted by $|a|$.
The symbol ε denotes set membership and $\subset$ denotes set inclusion.

R^n denotes an n-dimensional Euclidean space and the vector $x \varepsilon R^n$ is written $x = \text{col}[x_1,\ldots,x_n]$. The transpose of x is $x' = [x_1,\ldots,x_n]$ and the Euclidean norm of x is $|x| = \sqrt{x_1^2+\ldots+x_n^2}$. The set $J = [t_o,\infty)$, $t_o \geq 0$ and the set I denotes the sequence $\{t_o+k\}$, $k = 0,1,2,\ldots$. The notation $f\colon X \to Y$ refers to the mapping f from the set X into the set Y. The cartesian product of two sets X and Y is written as $X \times Y = \{(x,y)\colon x \varepsilon X \text{ and } y \varepsilon Y\}$.

For a matrix $A = (a_{ij})$, the transpose is denoted by A', the conjugate by $\bar{A}$ and the conjugate-transpose by A^*. $A \geq 0$ indicates $a_{ij} \geq 0$ for each pair (i,j). For a square matrix B, the determinant is denoted by $|B|$ and the inverse by B^{-1}. The eiganvalues are written as $\lambda(B)$. If all the eigenvalues are real, then the largest and the smallest eigenvalues are denoted by $\lambda_{max}(B)$ and $\lambda_{min}(B)$, respectively. The symbol I is used for the identity matrix.

If B is also symmetric, it is positive definite if $x'Bx > 0$ for all $x \varepsilon R^n$ and positive semidefinite if $x'Bx \geq 0$ for all $x \varepsilon R^n$. B is negative definite (semidefinite) if $x'Bx < 0$ ($x'Bx \leq 0$) for all $x \varepsilon R^n$.

The norm of a rectangular matrix A induced by the Euclidean norm is given as

$$||A|| = \min\{\alpha : \alpha|x| \geq |Ax|, \quad x \varepsilon R^n\} = \sqrt{\lambda_{max}(A^*A)}$$

CHAPTER 3. PREVIOUS RESULTS

The main objective of this chapter is to provide the necessary background material for this dissertation. Since the volume of work in the Lyapunov theory is enormous, only the very basic essentials are outlined here. An indication of previous work concerned with the stability analysis of composite systems is presented to show that the results in this dissertation were obtained in a similar manner. Finally, the well-known Popov criterion for absolute stability and its connection to Lyapunov theory is included because it is the main tool used to obtain the results in Chapter 4.

Lyapunov Theory

Systems are considered which may be described by ordinary differential equations of the form

$$\dot{x} = g(x,t) \tag{3.1}$$

where

$$\dot{x} = \frac{dx}{dt} \quad \text{and} \quad g: R^n x J \rightarrow R^n.$$

<u>Definition 3.1</u>: A function $g: R^n x J \rightarrow R^n$ is said to belong to <u>class E</u> if (i) for every $x_o \varepsilon R^n$ and for every $t_o \geq 0$, Equation (3.1) possesses one and only one solution $x(t; x_o, t_o)$ for all $t \varepsilon J$, where $x_o = x(t_o; x_o, t_o)$; and, (ii)

$g(x,t)=0$ for all $t\varepsilon J$ if and only if $x=0$ ($x=0$ is called the equilibrium of (3.1)).

Definition 3.2: The equilibrium $x=0$ of (3.1) is stable if for every $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon, t_o) > 0$ such that for all $x_o \varepsilon R^n$, $t_o \geq 0$, $|x_o|<\delta$ implies $|x(t; x_o, t_o)|<\varepsilon$ for all $t\varepsilon J$. The equilibrium is uniformly stable if the above δ is independent of t_o.

Definition 3.3: The equilibrium $x=0$ of (3.1) is asymptotically stable in the whole if for every $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon, t_o) > 0$ such that for all $x_o \varepsilon R^n$, $t_o \geq 0$, $|x_o|<\delta$ implies $|x(t; x_o, t_o)|<\varepsilon$ for all $t\varepsilon J$, and if for each $x_o \varepsilon R^n$, $\lim_{t\to\infty} |x(t; x_o, t_o)| = 0$. The equilibrium is uniformly asymptotically stable in the whole if the above δ is independent of t_o.

Definition 3.4: The equilibrium $x=0$ of (3.1) is uniformly exponentially stable in the whole if there exists $\alpha>0$ and $\beta>0$ such that $|x(t; x_o, t_o)| \leq \beta |_o| e^{-\alpha(t-t_o)}$ for all $x_o \varepsilon R^n$ and $t\varepsilon J$, where α, β depend on x_o.

Definition 3.5: The equilibrium $x=0$ of (3.1) is unstable if it is not stable.

Definition 3.6: The solutions of (3.1) are uniformly bounded if for every $\alpha>0$ there exists a $\beta=\beta(\alpha) > 0$ such that for all $x_o \varepsilon R^n$, $t_o \geq 0$, $|x_o|<\alpha$ implies $|x(t; x_o, t_o)|<\beta$ for all $t \varepsilon J$.

Results which yield conditions for stability in the sense of the above definitions involve the existence of mappings $V: R^n x J \to R^1$. Henceforth it is assumed that these mappings are continuous and continuously differentiable on $R^n x J$. The total derivative of V with respect to time along solutions of (3.1) is given by

$$DV_{(3.1)} = \nabla V(x,t)'g(x,t) + \frac{\partial V(x,t)}{\partial t}$$

where $\nabla V(x,t)$ denotes the gradient vector of the scalar function V.

Definition 3.7: A real-valued function $\phi(r)$ is said to belong to class K if it is defined, continuous and strictly increasing over $0 \leq r < \infty$ and if it vanishes at $r=0$.

Definition 3.8: A function $V(x,t)$ is said to be positive definite over $R^n x J$ if there exists a $\phi(r) \varepsilon K$ such that $V(x,t) \geq \phi(|x|)$ for all $t \varepsilon J$ and for all $x \varepsilon R^n$. If, in addition, $\lim_{r\to\infty} \phi(r) = \infty$, $V(x,t)$ is said to be radially unbounded. $V(x,t)$ is said to be negative definite over $R^n x J$ if $-V(x,t)$ is positive definite over $R^n x J$.

The next results yield sufficient conditions for stability in the sense of the above definitions (see, e.g., [8], [25]).

Theorem 3.1: The equilibrium $x=0$ of (3.1) is uniformly asymptotically stable in the whole if there exist a function $V:R^n xJ\rightarrow R^1$, two radially unbounded functions ϕ_1, $\phi_2 \varepsilon K$ and a function $\phi_3 \varepsilon K$ such that the conditions

(i) $\phi_1(|x|) \leq V(x,t) \leq \phi_2(|x|)$

(ii) $DV_{(3.1)} \leq -\phi_3(|x|)$

hold for all $x\varepsilon R^n$ and for all $t\varepsilon J$.

Theorem 3.2: The equilibrium $x=0$ of (3.1) is uniformly exponentially stable in the whole if there exist a function $V:R^n xJ\rightarrow R^1$ and three positive constants c_1, c_2, c_3 such that the conditions

(i) $c_1|x|^2 \leq V(x,t) \leq c_2|x|^2$

(ii) $DV_{(3.1)} \leq -c_3|x|^2$

hold for all $x\varepsilon R^n$ and for all $t\varepsilon J$.

Theorem 3.3: The equilibrium $x=0$ of (3.1) is unstable if there exists a function $V:R^n xJ\rightarrow R^1$ with the following properties.

(i) For each $\varepsilon>0$ and $t\varepsilon J$ there exist points $\bar{x}$ such that $V(\bar{x},t) < 0$ and $|\bar{x}|<\varepsilon$. The "domain $V<0$" is the set of all points (x,t) such that $|x|<h$ and $V(x,t)<0$ and is bounded by the hypersurfaces $|x|= h$ and $V=0$, where h is a constant.

(ii) In at least one of the component domains of the domain $V<0$, V is bounded from below and $DV_{(3.1)} \leq -\phi(V)$, $\phi \varepsilon K$.

<u>Theorem 3.4</u>: The solutions of (3.1) are uniformly bounded if there exist a function $V: R^n x J \to R^1$ and two radially unbounded functions ϕ_1, $\phi_2 \varepsilon K$ such that the conditions

(i) $\phi_1(|x|) \leq V(x,t) \leq \phi_2(|x|)$

(ii) $DV_{(3.1)} \leq 0$

hold for all $t \varepsilon J$ and $|x| \geq R$, where R is a constant.

Analogous definitions and theorems exist for systems described by difference equations of the form

$$x(\tau+1) = g[x(\tau), \tau], \quad g \varepsilon E \tag{3.2}$$

where

$g: R^n x I \to R^n$ and $g \varepsilon E$ is defined as before.

Conditions for the stability of (3.2) involve the existence of mappings $V: R^n x I \to R^1$. Similar theorems for stability, instability and boundedness exist for system (3.2). The conditions are analogous except that $t \varepsilon J$ is replaced by $\tau \varepsilon I$ and the total derivative is replaced by the first difference ΔV along solutions of (3.2), expressed by

$$\Delta V_{(3.2)} = V[g(x,\tau), \tau+1] - V(x,\tau).$$

Interconnected Systems

The results stated in the last section have been used extensively in the stability analysis of systems. However, they are severely limited in usefulness when applied to problems of high dimensions. For this reason it may be useful to view high-order systems as being composed of several lower-order (and hopefully simpler) subsystems which when interconnected in an appropriate fashion, yield the original higher order composite system. Using this approach, high-order composite systems can often then be analyzed in terms of their lower-order subsystems and in terms of their interconnecting structure.

In this dissertation, the systems considered are <u>composite systems</u> (or <u>interconnected systems</u>) which are defined as an interconnection of m <u>subsystems</u> (or <u>transfer systems</u>). The i<u>th</u> subsystem may be represented in a very general manner by the set of equations

$$\begin{aligned} \dot{z}_i &= f_i(z_i,t) + \ell_i(u_i,t) \\ y_i &= h_i(z_i,u_i,t) \end{aligned} \tag{3.3}$$

where z_i is an n_i-vector, u_i is a p_i-vector, $p_i \leq n_i$, and y_i is a q_i-vector and where $f_i: R^{n_i} \times J \to R^{n_i}$, $\ell_i: R^{p_i} \times J \to R^{n_i}$, $h_i: R^{n_i} \times R^{p_i} \times J \to R^{q_i}$. Equations (3.3) describe the input output characteristics of the ith subsystem S_i, where u_i is interpreted as the input and y_i as the output.

The m subsystems may be interconnected in a very general manner to form the composite system S according to the set of equations

$$u_i = g_i(y_1,\ldots,y_m,u,t) \;, \; i = 1,\ldots,m \qquad (3.4)$$

where u is a p-vector and $g_i: R^{q_1}x\ldots xR^{q_m}xR^p xJ \to R^{p_i}$. u is the input to the composite system. The ith subsystem together with its interconnections is shown in Figure 3.1.

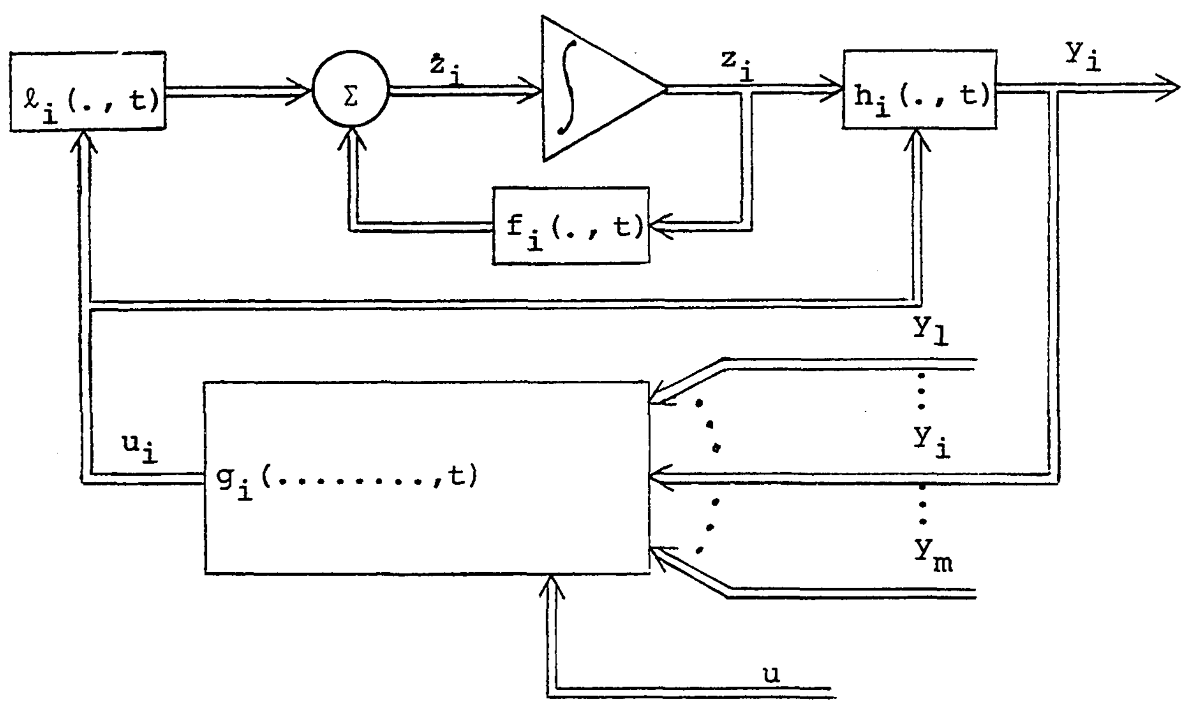

Figure 3.1. The ith subsystem (3.3) with its interconnections (3.4)

A simpler system can be described by changing the interconnecting equations such that the inputs u_i's are a function of the state variables instead of the output variables. This simplification does not diminish the generality of the representation because when a high-order system represented by its state equations is decomposed the interconnections are usually in terms of its state variables. Then the ith subsystem is represented by the equations

$$\dot{z}_i = f_i(z_i,t) + u_i$$
$$y_i = h_i(z_i,u_i,t) \qquad (3.5)$$

and the interconnections are represented by the equations

$$u_i = g_i(z_1,\ldots,z_m,u,t)\ ,\ i = 1,\ldots,m \qquad (3.6)$$

where g_i is the new mapping $g_i:R^{n_1}x\ldots xR^{n_m}xR^pxJ\to R^{p_i}$.
This representation is shown in Figure 3.2.

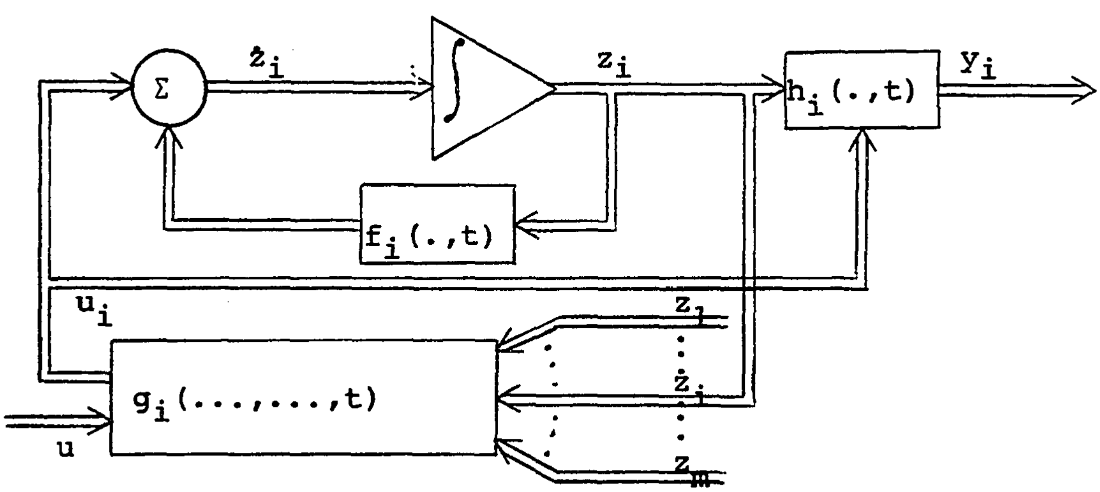

Figure 3.2. The ith subsystem (3.5) with interconnections (3.6)

Since the stability analysis of interest here deals with the state trajectories, the output equations become unnecessary. (However, when input-output stability is studied [20] the output functions are of extreme importance.)

The most common system that has been studied has the ith subsystem represented by

$$\dot{z}_i = f_i(z_i,t) + u_i \tag{3.7}$$

and the interconnections represented by

$$u_i = \sum_{j=1}^{m} g_{ij}(z_j,t) + g_{io}(u) \qquad i = 1,\ldots,m \tag{3.8}$$

where

$$f_i, g_{ij} \varepsilon E \quad \text{for } i = 1,\ldots,m \quad j = 0,1,\ldots,m.$$

Since the stability of interest is of the unforced system, that is, u=0, Equation (3.8) is usually written

$$u_i = \sum_{j=1}^{m} g_{ij}(z_j,t)\,,\ i = 1,\ldots,m \tag{3.9}$$

Since the term $g_{ii}(z_i,t)$ may be considered a part of the ith subsystem instead of an interconnection and included in $f_i(z_i,t)$, the Equation (3.9) is sometimes written

$$u_i = \sum_{\substack{j=1 \\ j\neq i}}^{m} g_{ij}(z_j,t)\,,\ i = 1,\ldots,m \tag{3.10}$$

Figure (3.3) shows such an interconnected system.

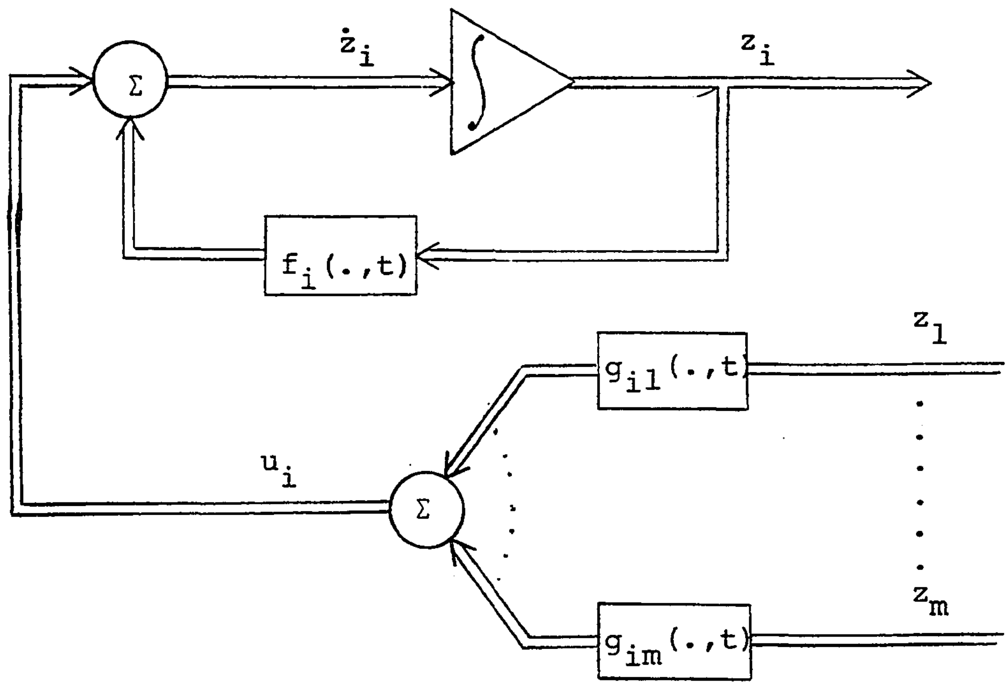

Figure 3.3. The ith subsystem (3.7) with interconnections (3.10)

If the interconnections are linear and time-invariant, Equation (3.10) may be written

$$u_i = \sum_{\substack{j=1 \\ j \neq i}}^{m} C_{ij} z_j \quad i = 1,\ldots,m \tag{3.11}$$

where C_{ij} is an $n_i x n_j$ constant matrix.

The system with the ith subsystem (3.7) and interconnections (3.10) may be written

$$\dot{z}_i = f_i(z_i,t) + \sum_{\substack{j=1 \\ j\neq i}}^{m} g_{ij}(z_j,t), \quad i = 1,\ldots,m \tag{3.12}$$

Letting $x' = [z_1' \ldots z_m']$, $f(x,t)' = (f_1(z_1,t)',\ldots,f_m(z_m,t)')$ and $g(x,t)' = ([\sum_{j=1}^{m} g_{1j}(z_j,t)]',\ldots, [\sum_{j=1}^{m} g_{mj}(z_j,t)]')$, system (3.12) may be written as

$$\dot{x} = f(x,t) + g(x,t) = h(x,t), \quad h\varepsilon E \tag{3.13}$$

where $h:R^n xJ\to R^n$, $f:R^n xJ\to R^n$, $g:R^n xJ\to R^n$. The system is then referred to as the composite system (3.13) with decomposition (3.12).

The system with linear time-invariant interconnections is referred to as the composite system

$$\dot{x} = f(x,t) + Cx = h(x,t), \quad h\varepsilon E, \tag{3.14}$$

with decomposition

$$\dot{z}_i = f_i(z_i,t) + \sum_{\substack{j=1 \\ j\neq i}}^{m} C_{ij}z_j, \quad i = 1,\ldots,m \tag{3.15}$$

where $h:R^n xJ\to R^n$, C is an nxn matrix and C_{ij} is a $n_i x n_j$ matrix, the i, jth partition of C.

The system described by the equation

$$\dot{z}_i = f_i(z_i,t), \quad f_i\varepsilon E \tag{3.16}$$

is known as the ith isolated subsystem of the composite system (3.12) or (3.14).

The next results yield sufficient conditions for stability of the above systems.

<u>Theorem 3.5</u>: (Bailey [5]) Assume that for each isolated subsystem (3.16) of composite system (3.14) there is a function $V_i(z_i,t)$ satisfying the conditions of Theorem 3.2 for positive constants c_{i1}, c_{i2} and c_{i3} and that there is a fourth positive constant c_{i4} such that $|\nabla V_i(z_i,t)| \leq c_{i4}|z_i|$ for all $t \varepsilon J$ and for all $z_i \varepsilon R^{n_i}$. Consider the $m\underline{th}$ order linear system

$$\dot{r} = Ar$$

where

$A = (a_{ij})$ is an mxm matrix with elements

$$a_{ij} = \begin{cases} -\dfrac{c_{i3}}{2c_{i2}}, & \text{if } i=j \\[2ex] \dfrac{c_{i4}^{\,2} \sum\limits_{j=1}^{m} ||C_{ij}||}{2c_{i3}c_{j1}}, & \text{if } i \neq j \text{ and } C_{ij} \neq 0 \\[2ex] 0, \text{ if } i \neq j \text{ and } C_{ij}=0 & \end{cases}$$

The equilibrium x=0 of composite system (3.14) is uniformly exponentially stable in the whole if the linear system $\dot{r}$ = Ar is asymptotically stable in the whole.

It should be remarked that the linear system $\dot{r}$=Ar is uniformly asymptotically stable in the whole if and only if all the eigenvalues of A have negative real parts. In the proof of Theorem 3.5 a vector Lyapunov function V with components

$V_i(z_i,t)$ is constructed and the vector inequality $\dot{V} \leq AV$ is used.

Theorem 3.6: (Michel [14]) The equilibrium $x=0$ of composite system (3.13) with decomposition (3.12) is uniformly exponentially stable in the whole if the following conditions are satisfied:

(i) each isolated subsystem (3.16) satisfies the conditions of Theorem 3.2 for positive constants c_{i1}, c_{i2} and c_{i3} and there exists a fourth positive constant c_{i4} such that $|\nabla V_i(z_i,t)| \leq c_{i4}|z_i|$ for all $t \in J$ and for all $z_i \in R^{n_i}$;

(ii) for each $i,j = 1,\ldots,m$, $i \neq j$, there exists a constant k_{ij} such that $|g_{ij}(z_j,t)| \leq k_{ij}|z_j|$ for all $t \in J$ and for all $z_j \in R^{n_j}$; and

(iii) there exists an m-vector $\alpha' = [\alpha_1,\ldots,\alpha_m]$ with $\alpha_i > 0$, $i = 1,\ldots,m$, such that the matrix $S = (s_{ij})$ defined by

$$s_{ij} = \begin{cases} -\alpha_i c_{i3}, & \text{if } i=j \\ \frac{1}{2}(\alpha_i c_{i4} k_{ij} + \alpha_j c_{j4} k_{ji}), & \text{if } i \neq j \end{cases}$$

is negative definite.

In the proof of this theorem a scalar Lyapunov function

$$V = \sum_{i=1}^{m} \alpha_i V_i(z_i,t)$$

is used. The theorem can be applied to composite system

(3.14) with decomposition (3.15) by ignoring condition (ii) and using $||C_{ij}||$ instead of k_{ij} for $i,j = 1,\ldots,m$, $i \neq j$. This result was shown to be better than that obtained in Theorem 3.5.

Theorem 3.7: (Michel [14]) The equilibrium $x=0$ of composite system (3.13) with decomposition

$$\dot{z}_i = f_i(z_i,t) + g_i(z_1,\ldots,z_m,t) \quad i = 1,\ldots,m \tag{3.17}$$

is asymptotically stable in the whole if the following conditions are satisfied:

(i) each isolated subsystem (3.16) satisfies the conditions of Theorem 3.1 for functions ϕ_{i1}, ϕ_{i2} and ϕ_{i3};

(ii) for each scalar product $\nabla V_i(z_i,t)'g_i(z_1,\ldots,z_m,t)$, $i = 1,\ldots,m$, an inequality of the form

$$\nabla V_i(z_i,t)'g_i(z_1,\ldots,z_m,t)$$

$$\leq [\phi_{i3}(|z_i|)]^{1/2} \sum_{j=1}^{m} a_{ij}(x,t)[\phi_{j3}(|z_j|)]^{1/2}$$

can be found for all $z_i \varepsilon R^{n_i}$, $z_j \varepsilon R^{n_j}$, $i,j=1,\ldots,m$ and for all $x \varepsilon R^n$, $t \varepsilon J$; and

(iii) there exists an m-vector $\alpha' = [\alpha_1,\ldots,\alpha_m]$, $\alpha_i > 0$, $i = 1,\ldots,m$, and $\varepsilon > 0$ such that for each $x \varepsilon R^n$ and each $t \varepsilon J$, the matrix $(S+\varepsilon I)$ is negative definite, where $S = (s_{ij})$ is defined by

$$s_{ij} = \begin{cases} -\alpha_i + \alpha_i a_{ii}(x,t), & \text{if } i=j \\ \frac{1}{2}[\alpha_i a_{ij}(x,t) + \alpha_j a_{j2}(x,t)], & \text{if } i \neq j \end{cases}.$$

The above theorems make up a small sample of the results obtained by many authors in the stability analysis of interconnected systems. Results on exponential stability are also obtained by Thompson [22] and on asymptotic stability and instability by Grujić and Siljak [7].

Similar results have been established for systems described by difference Equations [3], [6], [14] and also for systems described by functional differential Equations [14]. More recently, this work has been extended to stochastic systems [16] and to the analysis of bounded-input bounded-output stability [20].

Although none of these results are directly used in this dissertation, a sampling of these results are included here because this same general philosophy of interconnected systems is utilized throughout. The similarities in approach among these results and the results obtained in this dissertation are obvious.

The Popov Criterion

Systems are considered which may be described by equations of the form

$$\dot{x} = Ax - b\phi(\sigma)$$
$$\sigma = c'x \tag{3.18}$$

where x is an n-dimensional state vector, A is an nxn constant matrix, b and c are n-dimensional constant vectors, σ is a scalar variable and $\phi(\sigma)$ is an arbitrary, single-valued, piecewise continuous real function, defined for all values of σ and for a constant k, satisfying the conditions

$$\phi(0) = 0 \text{ and } 0 < \frac{\phi(\sigma)}{\sigma} \leq k, \quad \sigma \neq 0.$$

The pairs (A,b) and (A',c) are completely controllable. All the eigenvalues of A have negative real parts. This is known as the <u>principal case</u> of regulator systems [1] or the <u>direct control</u> problem [12].

If A is critical instead of stable such that it has a zero root and the rest of the roots are in the left half of the complex plane, system (3.18) may be described by

$$\dot{y} = \tilde{A}y - \tilde{b}\phi(\sigma)$$
$$\dot{\sigma} = \tilde{c}'y - \gamma\phi(\sigma) \tag{3.19}$$

where y is an (n-1)-vector (see [1]). This system is known as a <u>particular case</u> of regulator systems or the <u>indirect control</u> problem.

Definition 3.9: The system described by (3.18) or (3.19) is called absolutely stable in the sector (0, k) if the equilibrium is asymptotically stable in the whole and independent of the particular choice of the nonlinear function ϕ as long as this function satisfies the sector condition $0 < \frac{\phi(\sigma)}{\sigma} < k, \ \sigma \neq 0.$

To determine the absolute stability of system (3.18) Luré proposed the following Lyapunov function:

$$V = x'Px + \beta\int_0^\sigma \phi(\eta)d\eta \tag{3.20}$$

where P is an nxn symmetric matrix and β is a scalar. Then

$$DV_{(3.18)} = x'[A'P+PA]x - 2\phi[Pb- \tfrac{1}{2}\beta A'c]'x - \beta c'b\phi^2.$$

Adding and subtracting $\delta(\sigma- \frac{\phi}{k})\phi$ where δ is a scalar, and letting $\ell = \frac{1}{2}\beta A'c + \frac{1}{2}\delta c,$

$$\tau = \beta c'b + \frac{\delta}{k}$$

and

$$-Q = A'P + PA,$$

$$DV_{(3.18)} = -x'Qx - 2\phi(Pb-\ell)'x - \tau\phi^2 - \delta(\sigma- \tfrac{\phi}{k}) \ . \tag{3.21}$$

Then, for system (3.18) to be asymptotically stable in the whole, P and Q have to be positive definite, $\beta>0, \delta>0$ and $\tau > [Pb-\ell]'Q^{-1}[Pb-\ell]$.

Popov established a result using the frequency domain to determine absolute stability:

<u>Theorem 3.8</u>: System (3.18) is absolutely stable in the sector (0, k) if there exists a finite real number q such that for all $\omega \geq 0$

$$\mathrm{Re}(1+j\omega q)G(j\omega) + \frac{1}{k} > 0$$

where $G(j\omega) = c'(j\omega I-A)^{-1}b$, the frequency response of the linear part.

If a modified frequency response $G^*(\omega)$ is defined as

$$G^*(\omega) = \mathrm{Re}G(j\omega) + j\omega \mathrm{Im}G(j\omega) = X+jY$$

then the condition for absolute stability becomes

$$X-qY + \frac{1}{k} > 0 \quad \text{for all} \quad \omega \geq 0 .$$

Hence, the <u>Popov criterion</u> for the absolute stability in the sector (0, k) of system (3.18) is that the modified frequency response $G^*(\omega)$ lies strictly to the right of a straight line passing through the point $(-\frac{1}{k}, 0)$ in the $G^*(\omega)$ plane. This line has a slope 1/q and is known as the <u>Popov line</u> (see Figure 3.4).

This geometric criterion is a more powerful tool in determining stability than the Lyapunov method proposed by Luré because of its inherent simplicity. Although one method is simpler to use than the other, Kalman [11] and Yakubovich [24] showed that they are theoretically equivalent:

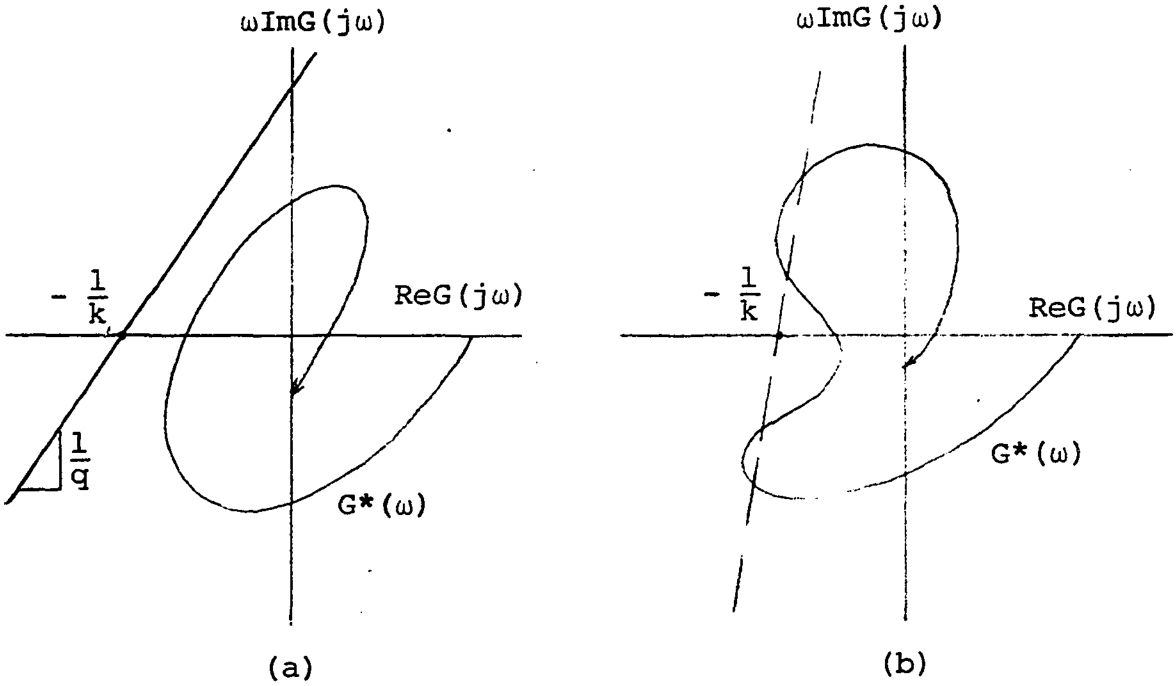

Figure 3.4. The modified frequency response showing (a) a stable case and (b) an unstable case

<u>Theorem 3.9</u>: For system (3.18) the Lyapunov function (3.20) is positive definite and radially unbounded and its derivative (3.21) is negative definite if and only if

$$\frac{\delta}{k} + \mathrm{Re}\{(\delta+j\omega\beta)c'(j\omega I-A)^{-1}b\} > 0 \text{ for all } \omega \geq 0$$

together with (i) $\delta>0$, $\beta>0$ and (ii) $\beta c'b + \frac{\delta}{k} > 0$ or

$\beta c'b + \frac{\delta}{k} = 0, \quad Pb - \frac{1}{2}\beta A'c - \frac{1}{2}\delta c = 0.$

The proof of this theorem is included in the Appendix. The main step in this proof is a lemma known as the Kalman-Yakubovich Lemma. A method to construct a Lyapunov function of the form (3.20) from the Popov line is outlined in the proof of this lemma.

The system described by (3.18) has one nonlinearity. Many authors [2], [9], [10], [17], [18] established results analogous to the Popov theorem for systems with multiple nonlinearities. Such a system may be described by the equations

$$\begin{aligned} \dot{x} &= Ax - B\phi(\sigma) \\ \sigma &= Hx \end{aligned} \tag{3.22}$$

where A is an nxn stable matrix, B is an nxm matrix and H is an mxn matrix, σ is an m-vector and $\phi: R^m \to R^m$. Certain authors [10], [17], [18] have considered $\phi(\sigma)' = [\phi_1(\sigma_1), \ldots, \phi_m(\sigma_m)]$ where $\phi_i: R^1 \to R^1$ with the sector conditions.

$$0 \le \phi_i(\sigma_i)\sigma_i \le k_i\sigma_i^2 \quad 0 \le k_i \le \infty \quad i = 1, \ldots, m \tag{3.23}$$

whereas Anderson [2] has used a more general form.

Theorem 3.10: (Narendra and Neuman [18]) System (3.22) with conditions (3.23) is absolutely stable if there exist diagonal matrices $\alpha \ge 0$, $\beta \ge 0$, $\alpha+\beta > 0$ such that

(i) $[\alpha K^{-1} + \beta HB + K^{-1}\alpha + B'H'\beta] \ge 0$, and

(ii) $[\alpha+\beta s][W(s)+K^{-1}] \geq 0$ for all s,

where K is a diagonal matrix with diagonal elements k_i and $W(s) = H'(sI-A)^{-1}B$.

Condition (ii) in this theorem is the multidimensional Popov criterion, which does not have a simple geometric interpretation. Like the Lyapunov method, this reduces to the finding of matrices that satisfy certain algebraic conditions. The advantage is that the matrices are of the same order as the number of nonlinearities which is usually less than the order of the system but the complexity of the method depends on the order of the transfer functions that are the elements of W(s).

CHAPTER 4. STABILITY FOR A CLASS OF SYSTEMS

In this chapter, a set of sufficient conditions for absolute stability of a class of systems with multiple nonlinearities are established. The procedure used is the same as that for interconnected systems outlined in the previous chapter. However, the class of systems discussed here can be decomposed in such a way that the stability of each isolated subsystem can be determined by using the simple Popov criterion. Hence, Lyapunov functions for the isolated subsystems can be constructed by using the Kalman-Yakubovich Lemma. The advantage in this method over the multidimensional Popov criterion in determining absolute stability is the usage of the simple Popov criterion which has a two-dimensional geometric interpretation in the frequency domain.

System Configuration

Consider the system described by the following equations

$$\left.\begin{aligned} \dot{z}_i &= A_i z_i - b_i \phi_i(\sigma_i) \\ y_i &= c_i' z_i \\ \sigma_i &= d_i' y = \sum_{j=1}^{m} d_{ij} y_j \end{aligned}\right\} \qquad i \varepsilon M = \{1,2,\ldots,m\} \tag{4.1}$$

where z_i is an n_i-dimensional state vector, A_i is an $n_i x n_i$ constant matrix, b_i and c_i are n_i-dimensional constant vectors, d_i is an m-dimensional constant vector, y_i and σ_i are scalars and $\phi_i:R^1 \to R^1$. In addition, A_i is a stable matrix, that is, all its eigenvalues have negative real parts, and $\phi_i(\sigma_i)$ is a piecewise continuous function of σ_i such that

$$\phi_i(0) = 0$$

and $0 < \dfrac{\phi_i(\sigma_i)}{\sigma_i} \leq k_i$, $\sigma_i \neq 0$, where k_i is a constant.

A portion of the block diagram of this system is shown in Figure 4.1. The purpose of the figure is to show one isolated subsystem with its interconnections.

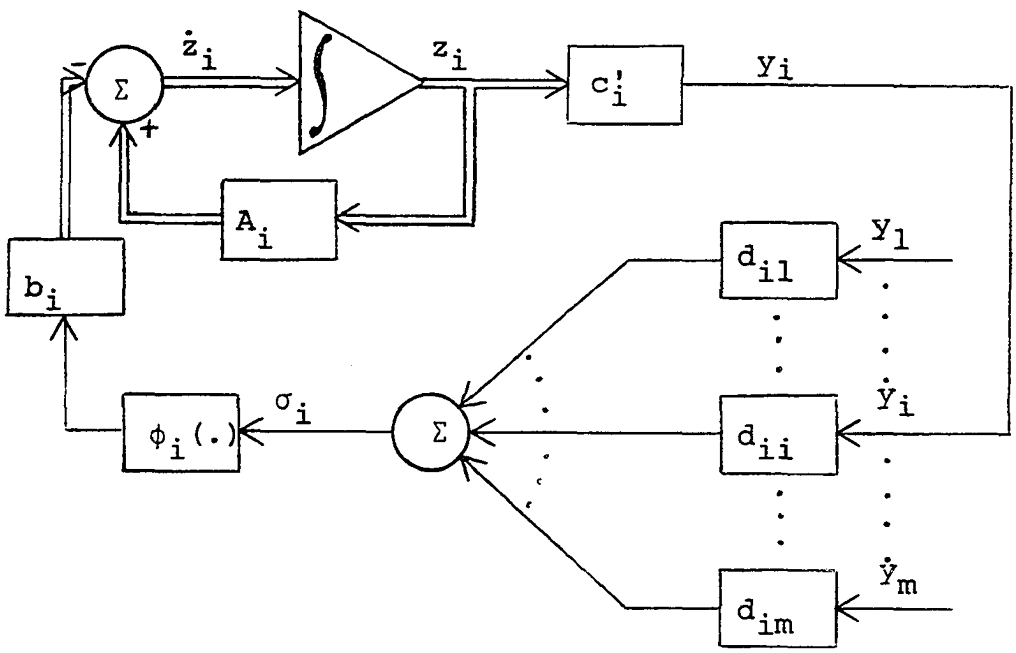

Figure 4.1. The ith subsystem and its interconnections of system (4.1)

The ith isolated subsystem can be written as

$$\begin{aligned} \dot{z}_i &= A_i z_i - b_i \phi_i(\sigma_i) \\ \sigma_i &= d_{ii}\, c_i' z_i \end{aligned} \qquad (4.2)$$

This is the familiar system that Lefschetz [12] calls the direct control problem and Aizerman and Gantmacher [1] call the principal case of regulator systems. This system can be represented by a simple nonlinearity $\phi_i(\sigma_i)$ and a linear transfer function $G_i(s)$ with a negative feedback where

$$G_i(s) = d_{ii} c_i' (sI - A_i) b_i$$

Figure 4.2 shows the isolated subsystem in this form with its interconnections. This figure is equivalent to Figure 4.1 but it shows more clearly that a combination of these isolated subsystems can be interconnected to form a large class of systems with multiple nonlinearities. Conversely, a large class of nonlinear systems can be decomposed into this type of subsystems and interconnections.

A very similar system can be described where the isolated subsystem represents the indirect control problem or a particular case of regulator systems. The system can be written as

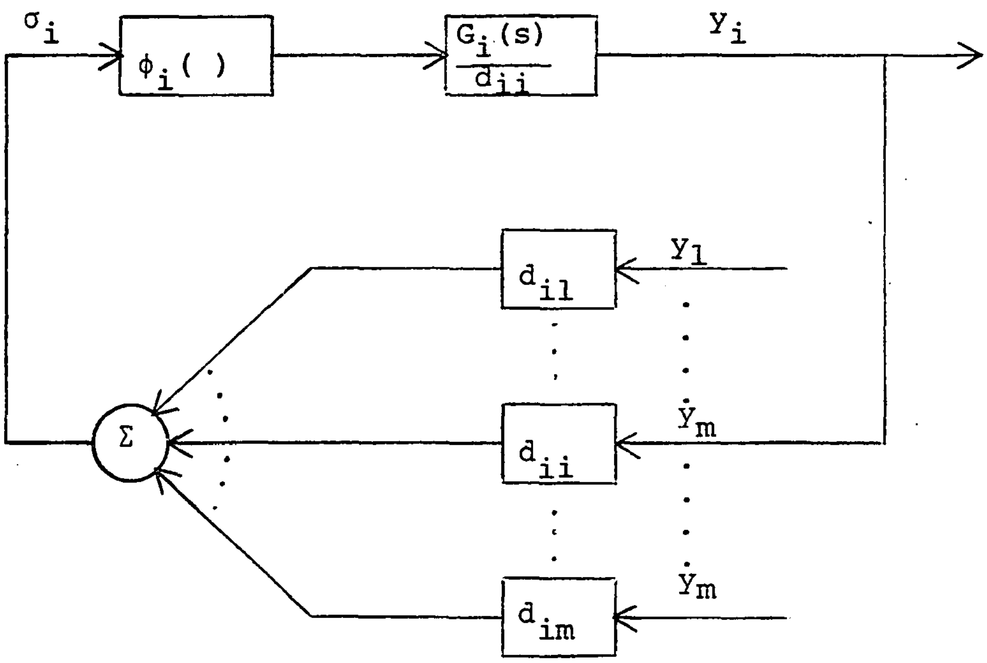

Figure 4.2. The ith subsystem in its transfer-function form of system (4.1)

$$\left.\begin{aligned} \dot{z}_i &= A_i z_i - b_i \phi_i(\sigma_i) \\ y_i &= c_i' z_i - \gamma_i \phi_i(\sigma_i) \\ \dot{\sigma}_i &= d_i' y = \sum_{j=1}^{m} d_{ij} y_j \end{aligned}\right\} \quad i \varepsilon M = \{1,2,\ldots,m\} \tag{4.3}$$

where all the symbols are the same as before and γ_i is a positive constant. The isolated subsystem is

$$\begin{aligned} \dot{z}_i &= A_i z_i - b_i \phi_i(\sigma_i) \\ \dot{\sigma}_i &= d_{ii}[c_i' z_i - \gamma_i \phi_i(\sigma_i)] \end{aligned} \tag{4.4}$$

As discussed in the previous chapter, an isolated subsystem like this is equivalent to subsystem (4.2) with A_i critical instead of stable such that all the eigenvalues of A_i in (4.2) have negative real parts except one which is zero. Subsystem (4.4) with its interconnections is shown in Figure 4.3.

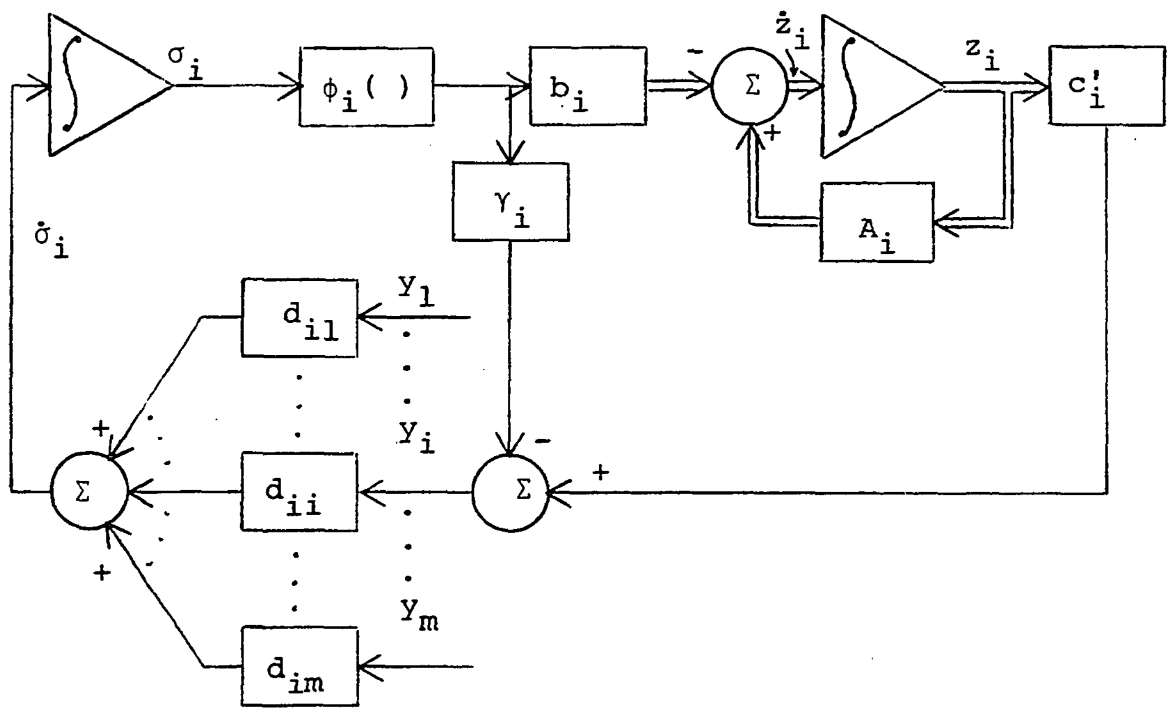

Figure 4.3. The ith subsystem and its interconnections of system (4.3)

The system configurations considered here are all described by equations that are continuous in time. The rest of this chapter is devoted to finding stability results for these types of systems. However, analogous results can be obtained for the systems of the same nature described by difference equations instead of differential equations. The discrete analog of the direct control problem is represented by

$$x(\tau+1) = Ax(\tau) - b\phi(\sigma(\tau))$$

$$\sigma(\tau) = c'x(\tau)$$

where $\tau \varepsilon I$ and the rest of the symbols are defined as before. The similarity with the continuous problem is obvious.

Stability Results

Consider the system (4.1) with isolated subsystems discussed by (4.2). Following Luré, the Lyapunov function for each subsystem may be chosen as

$$V_i = z_i' P_i z_i + \beta_i \int_0^{\sigma} \phi_i(\eta) d\eta$$

where P_i is a positive definite symmetric matrix and β_i is a positive constant.

Then the Lyapunov function for the whole system is

$$V = \sum_{i=1}^{m} \alpha_i V_i$$

$$= \sum_{i=1}^{m} \alpha_i z_i' P_i z_i + \sum_{i=1}^{m} \alpha_i \beta_i \int_0^{\sigma_i} \phi_i(\eta)\, d\eta$$

where $\alpha_1, \ldots, \alpha_m$ are positive constants. Then

$$\dot{DV}_{(4.1)} = \sum_{i=1}^{m} \alpha_i (A_i z_i - b_i \phi_i)' P_i z_i + \sum_{i=1}^{m} \alpha_i z_i' P_i (A_i z_i - b_i \phi_i)$$

$$+ \sum_{i=1}^{m} \alpha_i \beta_i \phi_i \sum_{j=1}^{m} d_{ij} c_j' (A_j z_j - b_j \phi_j)$$

$$= \sum_{i=1}^{m} \alpha_i z_i' (A_i' P_i + P_i A_i) z_i - 2 \sum_{i=1}^{m} \alpha_i b_i' P_i z_i \phi_i$$

$$+ \sum_{i=1}^{m} \alpha_i \beta_i \phi_i \sum_{j=1}^{m} d_{ij} c_j' A_j z_j$$

$$- \sum_{i=1}^{m} \alpha_i \beta_i \phi_i \sum_{j=1}^{m} d_{ij} c_j' b_j \phi_j$$

Adding and subtracting the nonnegative quantity $\sum_{i=1}^{m} \alpha_i (\sigma_i - \frac{\phi_i}{k_i}) \phi_i$ to $DV_{(4.1)}$ one obtains

$$DV_{(4.1)} = \sum_{i=1}^{m} \alpha_i z_i'(A_i'P_i+P_iA_i)z_i - 2\sum_{i=1}^{m} \alpha_i b_i'P_i z_i \phi_i$$

$$+ \sum_{i=1}^{m} \alpha_i \beta_i \phi_i \sum_{j=1}^{m} d_{ij} c_j' A_j z_j - \sum_{i=1}^{m} \alpha_i \beta_i \phi_i \sum_{j=1}^{m} d_{ij} c_j' b_j \phi_j$$

$$+ \sum_{i=1}^{m} \alpha_i \phi_i \sum_{j=1}^{m} d_{ij} c_j' z_j - \sum_{i=1}^{m} \alpha_i \frac{\phi_i^2}{k_i}$$

$$- \sum_{i=1}^{m} \alpha_i (\sigma_i - \frac{\phi_i}{k_i})\phi_i$$

$$= [z_1' \ldots z_m' \mid \phi_1 \ldots \phi_m] \left[\begin{array}{c|c} Q & S \\ \hline S' & R \end{array}\right] \begin{bmatrix} z_1 \\ \vdots \\ z_m \\ \hline \phi_1 \\ \vdots \\ \phi_m \end{bmatrix}$$

$$- \sum_{i=1}^{m} \alpha_i (\sigma_i - \frac{\phi_i}{k_i})\phi_i$$

where Q is an $\sum_{i=1}^{m} n_i \times \sum_{i=1}^{m} n_i$ matrix of the form

$$\begin{bmatrix} -\alpha_1 Q_1 & & 0 \\ & \ddots & \\ 0 & & -\alpha_m Q_m \end{bmatrix}$$ where each Q_i is an $n_i x n_i$ matrix and

$Q_i = -(A_i'P_i+P_iA_i)$, S is an $\sum_{i=1}^{m} n_i xm$ matrix of the form

$\begin{bmatrix} s_{11} & \cdots & s_{1m} \\ \vdots & & \vdots \\ s_{m1} & \cdots & s_{mm} \end{bmatrix}$ where each s_{ij} is an n_i-vector and

$$s_{ij} = \begin{cases} \frac{1}{2}A_i'c_id_{ii}\alpha_i\beta_i + \frac{1}{2}c_id_{ii}\alpha_i - P_ib_i\alpha_i, & \text{if } i=j \\ \frac{1}{2}A_i'c_id_{ji}\alpha_j\beta_j + \frac{1}{2}c_id_{ji}\alpha_j, & \text{if } i\neq j \end{cases}$$

and R is an mxm matrix with each element specified by

$$r_{ij} = \begin{cases} -\alpha_i\beta_id_{ii}c_i'b_i - \frac{\alpha_i}{k_i}, & \text{if } i=j \\ -\frac{1}{2}(\alpha_i\beta_id_{ij}c_j'b_j + \alpha_j\beta_jd_{ji}c_i'b_i), & \text{if } i\neq j \end{cases}$$

Then, system (4.1) is absolutely stable if the matrix $\left[\begin{array}{c|c} Q & S \\ \hline S' & R \end{array}\right]$ is negative definite.

It is interesting to note that this result is somewhat similar to that obtained by Lefschetz [12] for systems with multiple feedback. System (4.1) may be written in the form that Lefschetz uses:

$$\left.\begin{aligned} \dot{x} &= Ax - B\phi(\sigma) \\ y &= Cx \\ \sigma &= Dy \end{aligned}\right\} \tag{4.5}$$

where $x' = [z_1' \ldots z_m']$, $\phi(\sigma)$ is the m-vector with each element of the form $\phi_i(\sigma_i)$, A is an $\sum_{i=1}^{m} n_i \times \sum_{i=1}^{m} n_i$ matrix of the form

$$\begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{bmatrix},$$ B is an $\sum_{i=1}^{m} n_i x m$ matrix of the form

$$\begin{bmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_m \end{bmatrix},$$ C is an $m x \sum_{i=1}^{m} n_i$ matrix of the form

$$\begin{bmatrix} c_1' & & & 0 \\ & c_2' & & \\ & & \ddots & \\ 0 & & & c_m' \end{bmatrix}$$ and D is an mxm matrix of the form $[d_1 d_2 \ldots d_m]'$.

Choosing V(x), like Lefschetz, as

$$V(x) = x'Px + \int_0^{\sigma} \phi'(\eta)d\eta$$

one obtains

$$DV_{(4.5)} = x'(A'P+PA)x - \phi'(B'P- \frac{1}{2} DCA)x$$

$$- x'(PB- \frac{1}{2}A'C'D')\phi - \phi'DCB\phi$$

Adding and subtracting the nonnegative quantity $(\sigma'-\phi'K^{-1})\phi$

where K is an mxm diagonal matrix with each diagonal element k_i, one gets

$$DV_{(4.5)} = x'(A'P+PA)x - \phi'(B'P- \tfrac{1}{2}DCA- \tfrac{1}{2}C'D')x$$

$$- x'(PB- \tfrac{1}{2}A'C'D'- \tfrac{1}{2}DC)\phi$$

$$- \phi'(DCB+K^{-1})\phi - \sigma(\phi' - {}'K^{-1})\phi$$

$$= [x' \vdots \phi'] \left[\begin{array}{c|c} A'P+PA & -PB+\tfrac{1}{2}A'C'D'+\tfrac{1}{2}DC \\ \hline -B'P+\tfrac{1}{2}DCA+\tfrac{1}{2}C'D' & -DCB-K^{-1} \end{array}\right] \left[\begin{array}{c} x \\ \hline \phi \end{array}\right]$$

$$-[\sigma' - \phi'K^{-1}]\phi$$

If P is chosen to be of the form $\begin{bmatrix} P_1 & & & O \\ & P_2 & & \\ & & \ddots & \\ O & & & P_m \end{bmatrix}$, then the above partitioned matrix, which has to be negative definite to guarantee absolute stability of system (4.5), bears a striking resemblance to the partitioned matrix $\left[\begin{array}{c|c} Q & S \\ \hline S' & R \end{array}\right]$ which has to be negative definite for the absolute stability of system (4.1). Since Equations (4.1) and (4.5) represent the same system, the stability result obtained here is shown to be quite similar to that obtained by Lefschetz. However, the elements of the matrices Q, S and R are different from the matrices obtained by Lefschetz's method by factors that depend on the α_i's and β_i's, which are chosen arbitrarily. For this reason, there is more flexibility in the method that looks at the system in terms of its subsystems and interconnections and may possibly produce better results than the method used by Lefschetz.

A similar result can be obtained for system (4.3) with isolated subsystems (4.4). The Lyapunov function for each isolated subsystem is chosen again as

$$V_i = z_i' P_i z_i + \beta_i \int_0^{\sigma_i} \phi_i(\eta) d\eta$$

Then the Lyapunov function for the whole system is

$$V = \sum_{i=1}^{m} \alpha_i z_i' P_i z_i + \sum_{i=1}^{m} \alpha_i \beta_i \int_0^{\sigma_i} \phi_i(\eta) d\eta$$

and

$$DV_{(4.3)} = \sum_{i=1}^{m} \alpha_i z_i' (A_i' P_i + P_i A_i) z_i - 2 \sum_{i=1}^{m} \alpha_i z_i' P_i b_i \phi_i$$

$$+ \sum_{i=1}^{m} \alpha_i \beta_i \phi_i \sum_{j=1}^{m} d_{ij} c_j' z_j - \sum_{i=1}^{m} \alpha_i \beta_i \phi_i \sum_{j=1}^{m} d_{ij} \gamma_j \phi_j$$

$$= [z_1 \cdots z_m \vdots \phi_1 \cdots \phi_m] \left[\begin{array}{c|c} \tilde{Q} & \tilde{S} \\ \hline \tilde{S}' & \tilde{R} \end{array}\right] \begin{bmatrix} z_1 \\ \vdots \\ z_m \\ \hline \phi_1 \\ \vdots \\ \phi_m \end{bmatrix}$$

where the partitioned matrix is of the same form as before except that the n_i-vectors $\tilde{s}_{ij}$'s that constitute the matrix $\tilde{S}$

are given by

$$\tilde{s}_{ij} = \begin{cases} \frac{1}{2}\, c_i d_{ii} \alpha_i \beta_i - \alpha_i P_i b_i, & \text{if } i=j \\ \frac{1}{2}\, c_i d_{ji} \alpha_j \beta_j, & \text{if } i \neq j \end{cases}$$

and the elements $\tilde{r}_{ij}$'s of the matrix $\tilde{R}$ are given by

$$\tilde{r}_{ij} = \frac{1}{2}(\alpha_i \beta_i d_{ij} \gamma_j + \alpha_j \beta_j d_{ji} \gamma_i).$$

Again, this partitioned matrix has to be negative definite for system (4.3) to be absolutely stable.

Application of the Popov Criterion

The absolute stability of the isolated subsystem (4.2) can be determined by using the Popov criterion. If the Popov condition

$$\frac{\delta_i}{k_i} + \mathrm{Re}\{(\delta_i + j\omega\beta_i) G_i(j\omega)\} > 0 \tag{4.6}$$

where

$$G_i(j\omega) = d_{ii} c_i' (j\omega I - A_i) b_i$$

is satisfied by subsystem (4.2), then the subsystem is absolutely stable. Condition (4.6) has a geometric representation as shown in Chapter 3. The condition is satisfied if the modified frequency response

$$G_i^*(\omega) = \mathrm{Re}G_i(j\omega) + j\omega \mathrm{Im}G_i(j\omega)$$

lies strictly to the right of the Popov line passing through the point $-\frac{1}{k_i}$ of the real axis and having a slope of $\frac{\delta_i}{\beta_i}$.

By Theorem 3.9, the existence of a Popov line, that is, satisfaction of condition (4.6), is a necessary and sufficient condition for the existence of a positive definite Lyapunov function for subsystem (4.2) of the form

$$V_i = z_i' P_i z_i + \beta_i \int_0^{\sigma} \phi_i(\eta) d\eta$$

with a negative definite derivative

$$\begin{aligned} DV_{i(4.2)} = {} & z_i'(A_i'P_i + P_iA_i)z_i - 2\phi_i(P_ib_i - \tfrac{1}{2}\beta_i d_{ii} A_i' c_i \\ & - \tfrac{1}{2}\delta_i d_{ii} c_i)' z_i - (\beta_i d_{ii} c_i' b_i + \frac{\delta_i}{k_i})\phi_i^2 \\ & - \delta_i(\sigma_i - \frac{\phi_i}{k_i})\phi_i \end{aligned}$$

provided (i) $\delta_i > 0$, $\beta_i > 0$

and

(ii) $\beta_i d_{ii} c_i' b_i + \frac{\delta_i}{k_i} > 0$ or

$$\beta_i d_{ii} c_i' b_i + \frac{\delta_i}{k_i} = 0, \quad P_i b_i - \tfrac{1}{2}\beta_i d_{ii} A_i' c_i - \tfrac{1}{2}\delta_i d_{ii} c_i = 0$$

A method for the construction of the Lyapunov function V_i from the Popov line is outlined in the Appendix. The Lyapunov function V of the composite system (4.1) is taken to be a weighted sum of the m Lyapunov functions V_i of the m isolated subsystems (4.2) as in the previous section. The existence of such Lyapunov functions V_i are guaranteed by the existence of the Popov line for the subsystems. Hence, if the subsystems (4.2) satisfy the simple Popov criterion, the Lyapunov function V for the composite system (4.1) can be constructed. However, the absolute stability of composite system (4.1) still depends on the matrix $\left[\begin{array}{c|c} Q & S \\ \hline S' & R \end{array}\right]$ which can be found easily once V is constructed. The inherent difficulty in using the Direct Method of Lyapunov is in finding a suitable Lyapunov function and so the application of the Popov criterion makes it easier by providing a method to construct such a function for this class of systems.

This method to determine absolute stability for the class of systems described by Equations (4.1) should be compared to the multidimensional Popov criterion developed by several authors and discussed in Chapter 3. System (4.1) can be described by the Equations (4.5) and by Theorem 3.10 this system is absolutely stable if there exist diagonal matrices $\alpha \geq 0$, $\beta \geq 0$, $\alpha+\beta>0$ such that

$$\text{(i)} \quad (\alpha K^{-1} + \beta DCB + K^{-1}\alpha + B'C'D'\beta) \geq 0$$

and

$$\text{(ii)} \quad (\alpha+\beta s)(W(s)+K^{-1}) \geq 0$$

where W(s) = DC(sI-A)B and A, B, C, D, K are as defined earlier.

Condition (ii), unlike the simple geometric Popov criterion, is sometimes difficult to establish especially in the case of large systems. W(s) for system (4.5) written in terms of its subsystems is of the form

$$W(s) = \begin{bmatrix} d_{11}c_1'(sI-A_1)^{-1}b_1 & d_{12}c_2'(sI-A_2)^{-1}b_2 \cdots d_{1m}c_m'(sI-A_m)^{-1}b_m \\ d_{21}c_1'(sI-A_1)^{-1}b_1 & \vdots \\ \vdots & \\ d_{m1}c_1'(sI-A_1)^{-1}b_1 & \cdots \quad d_{mm}c_m'(sI-A_m)^{-1}b_m \end{bmatrix}$$

and so, finding the matrix $(\alpha+\beta s)$ to satisfy condition (ii) for a large system is usually difficult. It may be easier to establish the simple geometric Popov criterion for the m subsystems and constructing the partitioned matrix $\left[\begin{array}{c|c} Q & S \\ \hline S' & R \end{array}\right]$ which has to be negative definite.

Applications

<u>Example 4.1</u>: Consider the system

$$\begin{aligned} \dot{x} &= -ax - f(y) \\ \dot{y} &= -by - g(x) \end{aligned} \tag{4.7}$$

where $f(0)=0$, $g(0)=0$, $0<yf(y)\leq k_1y^2$ and $0<xg(x)\leq k_2x^2$. This can be written as

$$\left.\begin{aligned}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} &= \begin{bmatrix}-a & 0\\ 0 & -b\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} - \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}f(\sigma)\\ g(\xi)\end{bmatrix}\\ \begin{bmatrix}\sigma\\ \xi\end{bmatrix} &= \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}\end{aligned}\right\} \tag{4.8}$$

Narendra and Neuman [18] show how the multidimensional Popov criterion can be used to solve this system. For the interconnected system approach, this system may be written as consisting of the isolated subsystems

$$\dot{x} = -ax - f(\sigma) \quad \text{and} \quad \dot{y} = -by - g(\xi)$$

and the interconnections

$$\sigma = y \quad \text{and} \quad \xi = x$$

Let

$$V_1(x) = p_1x^2 + \beta_1\int_0^\sigma f(\eta)d\eta$$

$$V_2(y) = p_2y^2 + \beta_2\int_0^\xi g(\eta)d\eta$$

and

$V = \alpha_1V_1(x) + \alpha_2V_2(y)$. Then,

$$DV_{(4.8)} = [x\ y \mid f\ g] \left[\begin{array}{cc|cc} -2\alpha_1 a p_1 & 0 & -\alpha_1 p_1 & \frac{1}{2}-\frac{1}{2}\alpha_2\beta_2 a \\ 0 & -2\alpha_2 b p_2 & \frac{1}{2}-\frac{1}{2}\alpha_1\beta_1 b & -\alpha_2 p_2 \\ \hline -\alpha_1 p_1 & \frac{1}{2}-\frac{1}{2}\alpha_1\beta_1 b & -\frac{1}{k_1} & -\alpha_1\beta_1 \\ \frac{1}{2}-\frac{1}{2}\alpha_2\beta_2 a & -\alpha_2 p_2 & -\alpha_2\beta_2 & -\frac{1}{k_2} \end{array}\right] \left[\begin{array}{c} x \\ y \\ \hline f \\ g \end{array}\right]$$

$$-(\sigma - f/k_1)f - (\xi - g/k_2)g$$

Now α_1, α_2, β_1, β_2, p_1, p_2, which are all positive scalars can be chosen such that this matrix is negative definite. For instance, a choice of

$$\alpha_1\beta_1 = \frac{1}{b}, \quad \alpha_2\beta_2 = \frac{1}{a}, \quad \alpha_1 p_1 = \alpha_2 p_2 = \varepsilon > 0$$

yields the matrix

$$\begin{bmatrix} -2\varepsilon a & 0 & -\varepsilon & 0 \\ 0 & -2b\varepsilon & 0 & -\varepsilon \\ -\varepsilon & 0 & -\frac{1}{k_1} & -\frac{1}{b} \\ 0 & -\varepsilon & -\frac{1}{a} & -\frac{1}{k_2} \end{bmatrix}$$

which is negative definite if

$$a > \frac{1}{2}\varepsilon k_1 \quad \text{and}$$

$$\frac{ab}{k_1 k_2} - \frac{1}{2}\varepsilon\frac{a}{k_1} - \frac{1}{2}\varepsilon\frac{b}{k_1} > 1 - \frac{1}{4}\varepsilon^2$$

Hence, if $\varepsilon>0$ is chosen to be arbitrarily small, the condition for system (4.8) to be absolutely stable is

$$ab - k_1 k_2 > 0$$

This condition is the same as that obtained by the Routh-Hurwitz criterion on the linearized system. The same result was also obtained by Narendra and Neuman using the multidimensional Popov criterion.

Example 4.2: Consider a somewhat more complex system

$$\begin{aligned} \dot{x} &= -x - f(5x+2y) \\ \dot{y} &= -2y - g(x+5y) \end{aligned} \tag{4.9}$$

where $f(0)=0$, $0<\sigma f(\sigma)\leq k_1\sigma^2$, $g(0)=0$ and $0<\xi g(\xi)\leq k_2\xi^2$. This may be written as

$$\left.\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f(\sigma) \\ g(\xi) \end{bmatrix} \\ \begin{bmatrix} \sigma \\ \xi \end{bmatrix} &= \begin{bmatrix} 5 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}\right\} \tag{4.10}$$

In this case

$$W(s) + K^{-1} = \begin{bmatrix} \frac{5}{s+1} + \frac{1}{k_1} & \frac{2}{s+2} \\ \frac{1}{s+1} & \frac{5}{s+2} + \frac{1}{k_2} \end{bmatrix}$$

To use the multidimensional Popov criterion as shown by

Narendra and Neuman, the 2x2 diagonal positive matrices α and β have to be found such that $(\alpha+\beta s)(W(s)+K^{-1})$ is positive.

To use the interconnected systems approach the system can be considered to be made up of the isolated subsystems

$$\dot{x} = -x - f(\sigma) \quad \text{and} \quad \dot{y} = -2y - g(\xi)$$
$$\sigma = 5x \qquad\qquad \xi = 5y$$

and interconnected appropriately.

Let

$$V_1(x) = p_1x^2 + \beta_1\int_0^\sigma f(\eta)d\eta$$
$$V_2(y) = p_2y^2 + \beta_2\int_0^\xi g(\eta)d\eta$$

and

$$V = \alpha_1V_1(x) + \alpha_2V_2(y),$$

where p_1, p_2, β_1, β_2 are positive constants. Then

$$DV_{(4.10)}=[x\ y\ f\ g]$$
$$\begin{bmatrix} -2\alpha_1p_1 & 0 & -\frac{5}{2}\alpha_1\beta_1+\frac{5}{2}-\alpha_1p_1 & -\frac{1}{2}\alpha_2\beta_2+\frac{1}{2} \\ 0 & -4\alpha_2p_2 & -2\alpha_1\beta_1+1 & -\alpha_2\beta_2+\frac{1}{2}-p_2\alpha_2 \\ -\frac{5}{2}\alpha_1\beta_1+\frac{5}{2}-\alpha_1p_1 & -2\alpha_1\beta_1+1 & -5\alpha_1\beta_1 - \frac{1}{k_1} & -2\alpha_1\beta_1 \\ -\frac{1}{2}\alpha_2\beta_2+\frac{1}{2} & -\alpha_2\beta_2+\frac{1}{2}-p_2\alpha_2 & -\alpha_2\beta_2 & -5\alpha_2\beta_2-\frac{1}{k_2} \end{bmatrix}\begin{bmatrix} x \\ y \\ f \\ g \end{bmatrix}$$
$$-(\sigma-\frac{f}{k_1})f - (\xi-\frac{g}{k_2})g$$

Choosing $\alpha_1 p_1 = 1$, $\alpha_2 p_2 = 1$, $\alpha_1\beta_1 = \frac{1}{2}$, $\alpha_2\beta_2 = 1$, this matrix becomes

$$\begin{bmatrix} -2 & 0 & \frac{1}{4} & 0 \\ 0 & -4 & 0 & -\frac{3}{2} \\ \frac{1}{4} & 0 & -\frac{5}{2} - \frac{1}{k_1} & -1 \\ 0 & -\frac{3}{2} & -1 & -5 - \frac{1}{k_2} \end{bmatrix}$$

which is negative definite. It is also obvious that k_1 and k_2 do not affect the negative definiteness of this matrix. Hence system (4.10) is stable in the sector $(0, \infty)$ for both $f(\sigma)$ and $g(\xi)$.

Example 4.3: Consider the system

$$\left.\begin{aligned} \dot{z}_1 &= A_1 z_1 - b_1\phi_1(\sigma_1) \\ \sigma_1 &= c_1' z_1 + d_{12} c_2' z_2 \\ \dot{z}_2 &= A_2 z_2 - b_2\phi_2(\sigma_2) \\ \sigma_2 &= d_{21} c_1' z_1 + c_2' z_2 \end{aligned}\right\} \tag{4.11}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad d_{12} = \frac{1}{20}$$

and

and

$$A_2 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad d_{21} = \frac{1}{4}.$$

$$0 < \frac{\phi_1(\sigma_1)}{\sigma_1} \leq 7 \quad \text{and} \quad 0 < \frac{\phi_2(\sigma_2)}{\sigma_2} \leq \frac{3}{2}.$$

The first isolated subsystem is

$$\dot{z}_1 = A_1 z_1 - b_1 \phi_1(\sigma_1)$$
$$\sigma_1 = c_1' z_1$$

The transfer function for the linear part of this subsystem is

$$G_1(s) = \frac{1}{(s+1)^3}$$

and the application of the Popov criterion is shown in Figure (4.4(a)). The Popov line is chosen such that $\delta_1=1$, $\beta_1=1$. Then, the Lyapunov function V_1 is found by referring to the step by step construction procedure outlined in the Appendix:

$$\ell' = [0.5 \quad 0.5 \quad 0], \quad \tau = 0.143, \quad \nu = -0.125.$$

D is chosen to be I which gives $\mu=1$.

Then ε is chosen to be 0.018. This gives a q such that

$q' = [-0.684 \quad -0.21 \quad 0.479]$ from which one obtains

$$Q_1 = \begin{bmatrix} 0.483 & 0.143 & -0.328 \\ 0.143 & 0.059 & -0.1 \\ -0.328 & -0.1 & 0.244 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 1.002 & 1.233 & 0.241 \\ 1.233 & 1.664 & 0.421 \\ 0.241 & 0.421 & 0.181 \end{bmatrix}$$

Then $V_1 = z_1' P_1 z_1 + \beta_1 \int_0^{\sigma_1} \phi_1(\eta) d\eta$

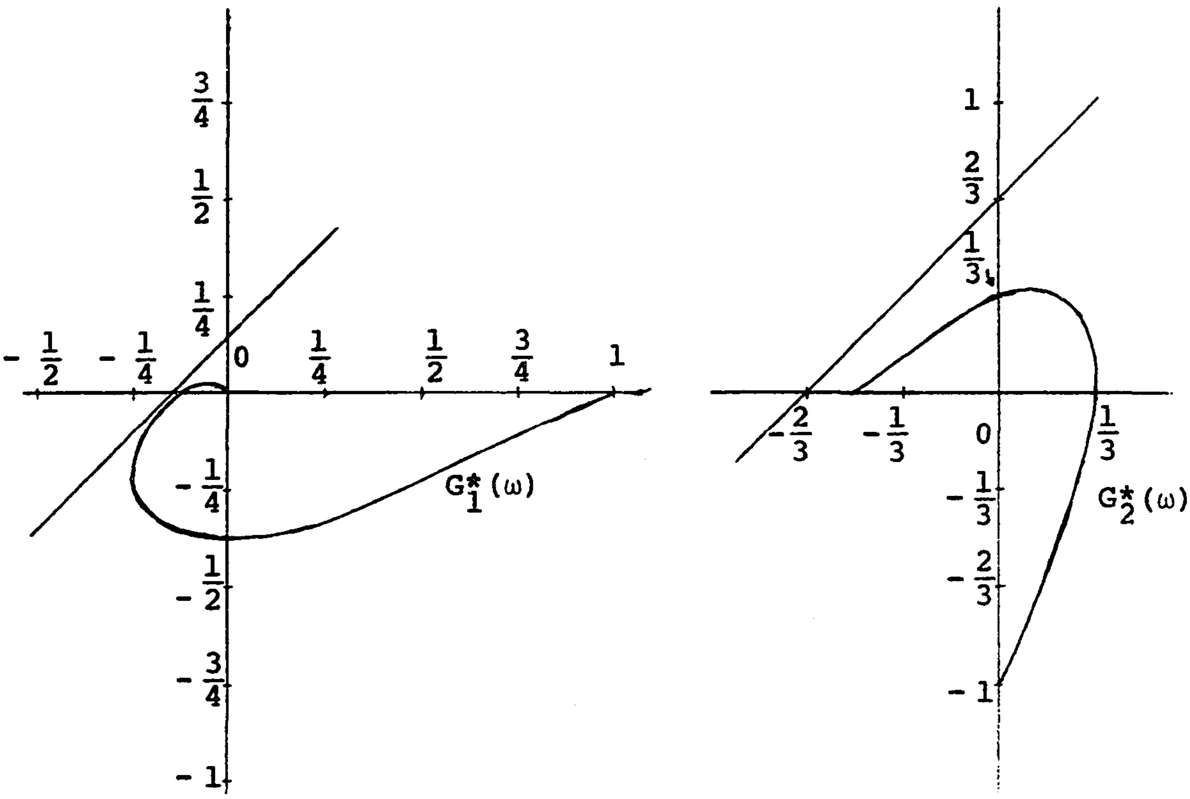

Figure 4.4. Popov criterion for the isolated subsystems of system (4.11)

The second isolated subsystem is

$$\dot{z}_2 = A_2 z_2 - b_2 \phi_2(\sigma_2)$$

$$\sigma_2 = c_2' z_2$$

which has the linear transfer function

$$G_2(s) = \frac{s-1}{(s+1)(s+2)}$$

By applying the Popov criterion, one obtains $\delta_1=1$, $\beta_1=1$.

If D is chosen to be I, then $\ell' = [-1.5 \quad -1.5]$, $\tau = 1.667$, $\nu = -1.5$, $\mu = 0.25$. Then ε is chosen to be 0.6 which gives $q' = [2.32 \quad 2.32]$. Then

$$Q_2 = \begin{bmatrix} 6 & 5.4 \\ 6 & 5.4 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 1.6 & 1.5 \\ 1.6 & 1.5 \end{bmatrix} \quad \text{and}$$

$$V_2 = z_2' P_2 z_2 + \beta_2 \int_0^{\sigma_2} \phi_2(\eta) d\eta \ .$$

The matrix $\left[\begin{array}{c|c} Q & S \\ \hline S' & R \end{array}\right]$ for this system with $\alpha_1 = 1$ and $\alpha_2 = 1$ then becomes

$$\left[\begin{array}{ccccc|cc}
-0.483 & -0.143 & 0.32 & & 0 & 0.259 & 0.125 \\
-0.143 & -0.059 & 0.1 & & 0 & 0.079 & 0.125 \\
0.328 & 0.1 & -0.244 & & 0 & -0.181 & 0 \\
0 & 0 & 0 & -6 & -6 & -0.075 & 0 \\
0 & 0 & 0 & -5.4 & -5.4 & -0.075 & 0 \\
\hline
0.259 & 0.079 & -0.181 & -0.074 & -0.075 & -0.143 & -0.025 \\
0.125 & 0.125 & 0 & 0 & 0 & -0.025 & -1.667
\end{array}\right]$$

The eigenvalues of this matrix are calculated to be -11.401, -1.697, -0.876, -0.6, -0.015, -0.007 and -0.0002 by a computer program using double precision. Hence, the matrix is negative definite and system (4.11) should be absolutely stable. However, the eigenvalues close to zero suggest that the system may be lightly damped.

CHAPTER 5. INSTABILITY AND BOUNDEDNESS

As discussed in Chapter 3, various authors have established sufficient conditions for the exponential and asymptotic stability of composite systems. In this chapter results are stated and proved which yield sufficient conditions for these systems to have unstable or bounded responses. The procedure employed in the proof of each result is similar to the procedure used by Michel [14]. The Lyapunov function for the composite system is obtained by a weighted summation of the Lyapunov functions for each isolated subsystem and the conditions for a particular type of system behavior is found from the requirements of the derivative of this composite Lyapunov function. Some examples of the direct application of these results are included.

Instability Theorems

Consider the system

$$\dot{x} = f(x,t) + g(x,t) = h(x,t) \tag{5.1}$$

where f, g and h belong to class E and are mappings from $R^n xJ$ into R^n. Let system (5.1) be decomposed as

$$\dot{z}_1 = f_i(z_i,t) + \sum_{\substack{j=1 \\ j\neq i}}^{m} g_{ij}(z_j,t), \quad i\varepsilon M = \{1,\ldots,m\} \tag{5.2}$$

where $f_i:R^{n_i}xJ\to R^{n_i}$, $g_{ij}:R^{n_j}xJ\to R^{n_i}$ and where $[z_1',\ldots,z_m'] = x'$. This is the same as system (3.13) with decomposition (3.12). The ith isolated subsystem is represented by

$$\dot{z}_i = f_i(z_i,t) \tag{5.3}$$

Theorem 5.1: If, for each isolated subsystem $\dot{z}_i = f_i(z_i,t)$, $i\varepsilon N\subset M$, there exists a Lyapunov function $V_i(z_i,t)$ with the following properties:

(i) $-c_{i1}|z_i|^2 \leq V_i(z_i,t) \leq -c_{i2}|z_i|^2$

(ii) $DV_{i(5.3)} \leq -c_{i3}|z_i|^2$

(iii) $|\nabla V_i(z_i,t)| \leq c_{i4}|z_i|$

for all z_i such that $|z_i| \leq h_i$ and all $t\varepsilon J$ where c_{i1}, c_{i2}, c_{i3}, c_{i4} and h_i are positive scalars; if, for each isolated subsystem $\dot{z}_i = f_i(z_i,t)$, $i\notin N$, $i\varepsilon M$, there exists a Lyapunov function $V_i(z_i,t)$ with the following properties:

(i) $c_{i1}|z_i|^2 \leq V_i(z_i,t) \leq c_{i2}|z_i|^2$

(ii) $DV_{i(5.3)} \leq -c_{i3}|z_i|^2$

(iii) $|\nabla V_i(z_i,t)| \leq c_{i4}|z_i|$

for all z_i such that $|z_i| \leq h_i$ and all $t\varepsilon J$ where c_{i1}, c_{i2}, c_{i3}, c_{i4} and h_i are positive scalars; if $|g_{ij}(z_j,t)| \leq k_{ij}|z_j|$ for all z_j such that $|z_j| \leq h_j$, for all $t\varepsilon J$ and for all $i, j\varepsilon M$, $i\neq j$, where k_{ij} are positive scalars; and if there exist $\alpha_i>0$, $i\varepsilon M$ such that the matrix $S = (s_{ij})$ defined by

$$s_{ij} = \begin{cases} -\alpha_i c_{i3}, & \text{if } i=j \\ \frac{1}{2}(\alpha_i c_{i4} k_{ij} + \alpha_j c_{j4} k_{ji}), & \text{if } i\neq j \end{cases}$$

is negative definite; then the equilibrium $x=0$ of composite system (5.1) with decomposition (5.2) is unstable.

Proof: Let $V(x,t) = \sum_{i=1}^{m} \alpha_i V_i(z_i,t)$. If $z_i = 0$ for $i\notin N$ and $|z_i| \leq h_i$ for $i\varepsilon N$, then

$$\sum_{i\varepsilon N} -\alpha_i c_{i1}|z_i|^2 \leq V(x,t) = \sum_{i\varepsilon N} \alpha_i V_i(z_i,t) \leq \sum_{i\varepsilon N} -\alpha_i c_{i2}|z_i|^2,$$

that is, $V(x,t)$ is negative and bounded from below in a domain bounded by the hypersurfaces $|z_i| = h_i$, $i\varepsilon N$ and $V = 0$. Now,

$$\begin{aligned} DV_{(5.1)} &= \sum_{i=1}^{m} \alpha_i DV_{i(5.3)} + \sum_{i=1}^{m} \alpha_i \nabla V_i'(z_i,t) \sum_{\substack{j=1 \\ j\neq i}}^{m} g_{ij}(z_j,t) \\ &\leq \sum_{i=1}^{m} -\alpha_i c_{i3}|z_i|^2 + \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j\neq i}}^{m} \alpha_i c_{i4}|z_i||z_j| \end{aligned}$$

in the domain where $|z_i| \leq h_i$ for $i\varepsilon M$. Let $y' = [|z_1|\ldots|z_m|]$ and let $R = (r_{ij})$ be a matrix such that

$$r_{ij} = \begin{cases} -\alpha_i c_{i3}, & \text{if } i=j \\ \alpha_i c_{i4} k_{ij}, & \text{if } i\neq j \end{cases}$$

Then,

$$DV_{(5.1)} \leq y'Ry = \tfrac{1}{2}y'(R+R')y = y'Sy$$

Since S is symmetric and negative definite, $\lambda_{max}(s)$ is negative and $DV_{(5.1)} \leq \lambda_{max}(S)|y|^2 = \lambda_{max}(S)|x|^2$ in a neighborhood of the origin, $|z_i| \leq h_i$, $i\varepsilon M$.

Therefore, by Theorem 3.3, the equilibrium x=0 of system (5.1) is unstable.

Theorem (5.1) establishes the sufficient conditions for instability of the composite system when one or more of the subsystems are unstable. In the hypotheses, the stable subsystems are assumed to be exponentially stable in a neighborhood of the equilibrium point. Other authors have shown, as in Theorems 3.6 and 3.7, that composite system (5.1) is stable if all its isolated subsystems are stable. Theorem (5.1) shows that the system may be unstable if only one of the subsystems is unstable.

In the above hypotheses the Lyapunov functions $V_i(z_i, t)$ for the isolated subsystems are bounded by the functions $c_{i1}|z_i|^2$ and $c_{i2}|z_i|^2$ or their negatives. These bounds may be replaced by more general radially unbounded functions $\phi_{i1}(|z_i|)$ and $\phi_{i2}(|z_i|)$, ϕ_{i1}, $\phi_{i2}\varepsilon K$, without any change in result.

This theorem can be applied to composite systems with linear interconnections. Consider the system

$$\dot{x} = f(x,t) + Cx = h(x,t),\ h\varepsilon E \tag{5.4}$$

where C is an nxn matrix, with decomposition

$$\dot{z}_i = f_i(z_i, t) + \sum_{\substack{j=1 \\ j \neq i}}^{m} C_{ij} z_j, \quad i \varepsilon M \tag{5.5}$$

This system is the same as system (3.14) with decomposition (3.15) and its ith isolated subsystem is represented by (5.3).

Corollary 5.1: If, for each isolated subsystem $\dot{z}_i = f_i(z_i, t)$, $i \varepsilon N \subset M$, there exists a Lyapunov function $V_i(z_i, t)$ with the following properties:

(i) $-c_{i1}|z_i|^2 \leq V_i(z_i, t) \leq -c_{i2}|z_i|^2$

(ii) $DV_{i(5.3)} \leq -c_{i3}|z_i|^2$

(iii) $|\nabla V_i(z_i, t)| \leq c_{i4}|z_i|$

for all z_i such that $|z_i| \leq h_i$ and all $t \varepsilon J$, where c_{i1}, c_{i2}, c_{i3}, c_{i4} and h_i are positive scalars; if, for each isolated subsystem $\dot{z}_i = f_i(z_i, t)$, $i \notin N$, $i \varepsilon M$, there exists a Lyapunov function $V_i(z_i, t)$ with the following properties:

(i) $c_{i1}|z_i|^2 \leq V_i(z_i, t) \leq c_{i2}|z_i|^2$

(ii) $DV_{i(5.3)} \leq -c_{i3}|z_i|^2$

(iii) $|\nabla V_i(z_i, t)| \leq c_{i4}|z_i|$

for all z_i such that $|z_i| \leq h_i$ and all $t \varepsilon J$, where c_{i1}, c_{i2},

c_{i3}, c_{i4} and h_i are positive scalars; and if there exist $\alpha_i>0$, $i\varepsilon M$, such that the matrix $S = (s_{ij})$ defined by

$$s_{ij} = \begin{cases} -\alpha_i c_{i3}, & \text{if} \quad i=j \\ \frac{1}{2}\,(\alpha_i c_{i4}||C_{ij}|| + \alpha_j c_{j4}||C_{ji}||), & \text{if} \quad i\neq j \end{cases}$$

is negative definite; then, the equilibrium $x=0$ of composite system (5.4) with decomposition (5.5) is unstable.

Proof: The proof is identical with the proof of Theorem 5.1 with k_{ij} replaced by $||C_{ij}||$.

Similar results can be obtained for other types of system configurations using similar procedures (see [14]). Instead of doing that the next theorem deals with systems described by difference equations. Consider the system

$$x(\tau+1) = f[x(\tau),\tau] + g[x(\tau),\tau] = h[x(\tau),\tau] \tag{5.6}$$

where f, g and h are in class E and are mappings from $R^n xI$ to R^n, with decomposition

$$z_i(\tau) = f_i[z_i(\tau),\ \tau] + \sum_{\substack{j=1 \\ j\neq 1}}^{m} g_{ij}[z_j(\tau),\ \tau], \quad i = 1,\ldots,m \tag{5.7}$$

where $f_i:R^{n_i}xI\to R^{n_i}$, $g_{ij}:R^{n_j}xI\to R^{n_i}$ and $[z_1',\ldots,z_m'] = x'$. The ith isolated subsystem can then be represented by

$$z_i(\tau+1) = f_i[z_i(\tau),\ \tau] \tag{5.8}$$

The ith isolated subsystem and its interconnections are shown in Figure (5.1).

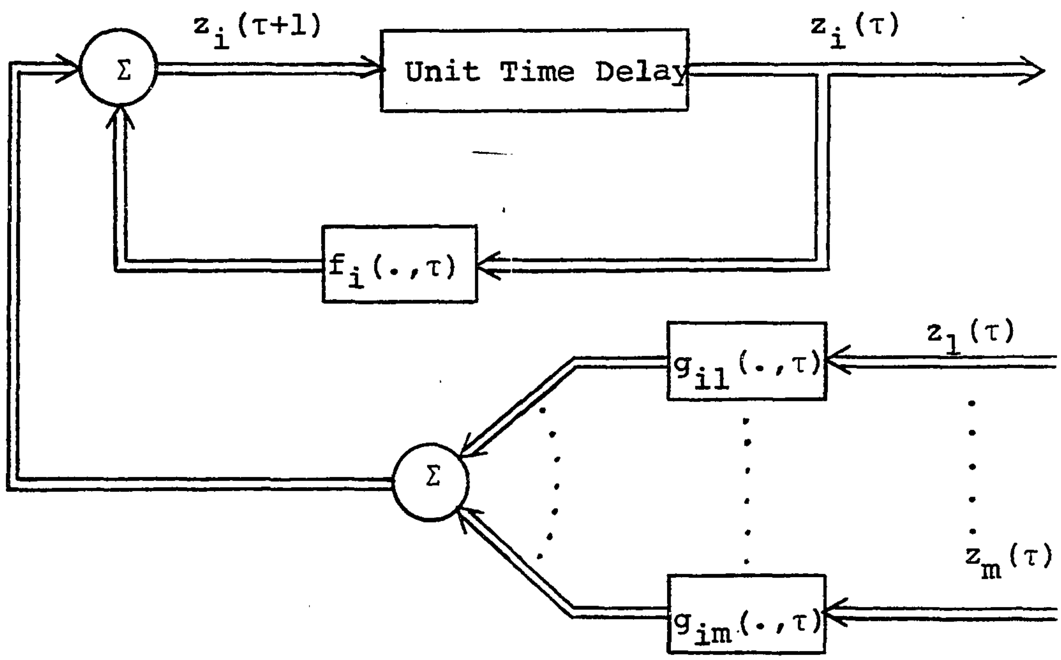

Figure 5.1. The ith subsystem of system (5.6) and its interconnections

<u>Theorem 5.2</u>: If, for each isolated subsystem $z_i(\tau+1) = f_i[z_i(\tau), \tau]$, $i\varepsilon NCM$, there exists a Lyapunov function $V_i(z_i, \tau)$ with the following properties.

(i) $-\phi_{i1}(|z_i|) \leq V_i(z_i, \tau) \leq -\phi_{i2}(|z_i|)$

(ii) $\Delta V_{i(5.8)} \leq -c_i|z_i|$

(iii) $|V_i(\bar{z}_i, \tau)-V_i(z_i, \tau)| \leq L_i|\bar{z}_i-z_i|$

for all z_i, $\bar{z}_i$ such that $|z_i| \leq h_i$, $|\bar{z}_i| \leq h_i$ and all $\tau\varepsilon I$ where ϕ_{i1}, $\phi_{i2}\varepsilon K$ and c_i, L_i are positive scalars; if, for each isolated subsystem $z_i(\tau+1) = f_i[z_i(\tau), \tau]$, $i \not\in N$, $i\varepsilon M$, there exists a Lyapunov function $V_i(z_i, \tau)$ with the following properties:

(i) $\phi_{i1}(|z_i|) \leq V_i(z_i, \tau) \leq \phi_{i2}(|z_i|)$

(ii) $\Delta V_{i(5.8)} \leq -c_i|z_i|$

(iii) $|V_i(\bar{z}_i, \tau) - V_i(z_i, \tau)| \leq L_i|\bar{z}_i - z_i|$

for all z_i, $\bar{z}_i$ such that $|z_i| \leq h_i$, $|\bar{z}_i| \leq h_i$ and all $\tau\varepsilon I$ where ϕ_{i1}, $\phi_{i2}\varepsilon K$ and c_i, L_i are positive scalars; if $|g_{ij}(z_j, \tau)| \leq k_{ij}|z_j|$ for all z_j such that $|z_j| \leq h_j$, for all $\tau\varepsilon I$ and for all $i,j\varepsilon M$, $i \neq j$, where k_{ij} are positive scalars; and if there exists a matrix $S = (s_{ij})$ defined by

$$s_{ij} = \begin{cases} c_i, & \text{if } i=j \\ -L_i k_{ij}, & \text{if } i \neq j \end{cases}$$

with all its successive principal minors positive; then the equilibrium $x=0$ of composite system (5.6) with decomposition (5.7) is unstable.

<u>Proof</u>: Let $\alpha' = [\alpha_1 \ldots \alpha_m]$ be an arbitrary vector with $\alpha_i > 0$, $i = 1,\ldots,m$, and let $V(x,\tau) = \sum_{i=1}^{m} \alpha_i V_i(z_i, \tau)$. If $z_i = 0$ for $i \not\in N$, then

$$-\sum_{i\varepsilon N}\alpha_i\phi_{i1}(|z_i|) \leq V(x,\tau) = \sum_{i\varepsilon N}\alpha_i V_i(z_i,\ \tau)$$

$$\leq -\sum_{i\varepsilon N}\alpha_i\phi_{i2}(|z_i|)$$

that is, $V(x,\tau)$ is negative and bounded in the domain bounded by the hypersurfaces $|z_i| = h_i$ and $V=0$. Then

$$\Delta V_{i(5.7)} = \sum_{i=1}^{m}\alpha_i V_i[f_i(z_i,\tau) + \sum_{\substack{j=1\\j\neq i}}^{m} g_{ij}(z_j,\tau),\ \tau+1]$$

$$-\sum_{i=1}^{m}\alpha_i V_i[z_i,\tau]$$

$$= \sum_{i=1}^{m}\alpha_i\{V_i[f_i(z_i,\tau)+\sum_{\substack{j=1\\j\neq 1}}^{m} g_{ij}(z_j,\tau),\tau+1]$$

$$- V_i[f_i(z_i,\tau),\tau+1]\}$$

$$+ \sum_{i=1}^{m}\alpha_i\{V_i[f_i(z_i,\tau),\tau+1]-\alpha_i V_i[z_j,\tau]\}$$

$$\leq \sum_{i=1}^{m}\alpha_i L_i \sum_{\substack{j=1\\j\neq i}}^{m}|g_{ij}(z_j,\tau)| - \sum_{i=1}^{m}\alpha_i c_i|z_i|$$

$$\leq \sum_{i=1}^{m}\sum_{\substack{j=1\\j\neq i}}^{m}\alpha_i L_i k_{ij}|z_j| - \sum_{i=1}^{m}\alpha_i c_i|z_i|$$

Let $y' = [|z_1|,\ldots,|z_2|]$. Then the above inequality may be written as

$$\Delta V_{(5.7)} \leq -\alpha' S y$$

Since S has positive principal minors and $s_{ij} \leq 0$ for all $i \neq j$, S^{-1} exists and all elements of S^{-1} are nonnegative [14]. Hence, α can always be chosen such that $\alpha' S > 0$. Then $\Delta V_{(5.7)} < 0$ for $x \neq 0$ and $\Delta V_{(5.7)} = 0$, $x=0$, for all $\tau \in I$. Therefore, the equilibrium $x=0$ of system (5.6) is unstable.

Applications

<u>Example 5.1</u>: Consider the system

$$\left.\begin{aligned}
\dot{x}_1 &= 2x_1 + 2x_2 \\
\dot{x}_2 &= x_1 + 3x_2 \\
\dot{x}_3 &= g_1(x_1) + a(t)x_4 + g_2(x_6) \\
\dot{x}_4 &= g_3(x_2) - x_3 - 2a(t)x_4 \\
\dot{x}_5 &= x_2 + \sqrt{3}\, x_4 - 20x_5 + x_6^2 \\
\dot{x}_6 &= x_2 + x_3 - x_5x_6 - 25x_6 - x_5x_7 \\
\dot{x}_7 &= b(t)x_1 + x_3 + x_5x_6 - 20x_7
\end{aligned}\right\} \tag{5.9}$$

where $g_1(x_1) \leq \frac{1}{4} x_1$, $a(t)$ is a continuous function such that $a^{-1}(t)$ exists for all $t \in J$ and $\frac{1}{2} \leq a^{-1}(t) \leq 1$ and $0 \leq \frac{da^{-1}(t)}{dt} \leq 1$, $g_2(x_6) \leq x_6$, $g_3(x_2) \leq \frac{1}{4} x_2$, $b(t) \leq 1$ for all $t \in J$.

Let the first isolated subsystem be

$$\dot{z}_1 = A_1 z_1 \text{ where } z_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \text{ and}$$

$V_1 = z_1' P_1 z_1$ where $P_1 = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$. Then, $DV_1 = z_1' Q_1 z_1$ where $Q_1 = \begin{bmatrix} -6 & 0 \\ 0 & -2 \end{bmatrix}$ and

(i) $-2.62|z_1|^2 \le V_1 \le -0.38|z_1|^2$

(ii) $DV_1 \le -2|z_1|^2$

(iii) $|\nabla V_1| \le 5.24|z_1|$.

This is obviously an unstable subsystem.

Let the second isolated subsystem be

$$\dot{z}_2 = A_2(t) z_2 \quad \text{where} \quad z_2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \text{ and } A_2(t) = \begin{bmatrix} 0 & a(t) \\ -1 & -2a(t) \end{bmatrix}$$

and $V_2 = z_2' P_2(t) z_2$ where $P_2(t) = \begin{bmatrix} 2+a^{-1}(t) & 1 \\ 1 & 1 \end{bmatrix}$

Then $DV_2 = z_2' Q_2(t) z_2$ where $Q_2(t) = \begin{bmatrix} -2 & 0 \\ 0 & -4a(t) \end{bmatrix}$

and (i) $\frac{1}{2}|z_2|^2 \le V_2 \le \frac{7}{2}|z_2|^2$

(ii) $DV_2 \le -2|z_2|^2$

(iii) $|\nabla V_2| \le 7|z_2|$

This isolated subsystem is stable.

Let the third isolated subsystem be

$$\dot{z}_3 = f(z_3) \text{ where } z_3 = \begin{bmatrix} x_5 \\ x_6 \\ x_7 \end{bmatrix} \text{ and } f(z_3) = \begin{cases} -20x_5 + x_6^2 \\ -x_5x_6 - 25x_6 - x_5x_7 \\ x_5x_6 - 20x_7 \end{cases}$$

and

$$V_3 = \frac{1}{2} z_3' z_3 \ .$$

Then

$$DV_3 = z_3' f(z_3) = -20x_5^2 - 25x_6^2 - 20x_7^2$$

and

(i) $\frac{1}{2} |z_3|^2 \leq V_3 \leq \frac{1}{2} |z_3|^2$

(ii) $DV_3 \leq -20|z_3|^3$

(iii) $|\nabla V_3| \leq |z_3|$

This isolated subsystem is also stable.

The interconnections between the subsystems are given by (using the notation of Equation (5.2))

$$g_{12}(z_2) = 0$$

$$g_{13}(z_3) = 0$$

$$g_{21}(z_1) = \begin{bmatrix} g_1(x_1) \\ g_3(x_2) \end{bmatrix} \quad \text{where } g_1(x_1) \leq \frac{1}{4} x, \text{ and } g_3(x_2) \leq \frac{1}{4} x_2$$

$$g_{23}(z_1) = \begin{bmatrix} g_2(x_6) \\ 0 \end{bmatrix} \text{ where } g_2(x_6) \leq x_6$$

$$g_{31}(z_1) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ b(t) & 0 \end{bmatrix} z_1 \quad \text{where} \quad b(t) \leq 1 \quad \text{for all} \quad t\varepsilon J$$

$$g_{32}(z_2) = \begin{bmatrix} 0 & \sqrt{3} \\ 1 & 0 \\ 1 & 0 \end{bmatrix} z_2$$

and $k_{12} = k_{13} = 0$, $k_{21} = \frac{1}{4}$, $k_{23} = 1$, $k_{31} = 2$, $k_{32} = 3$ satisfy the conditions $|g_{ij}(z_j)| \leq k_{ij}|z_j|$ for $i,j = 1,2,3$, $i \neq j$.

Then using $\alpha_1 = \alpha_2 = \alpha_3 = 1$, one obtains

$$S = \begin{bmatrix} -2 & \frac{7}{8} & 1 \\ \frac{7}{8} & -2 & 5 \\ 1 & 5 & -20 \end{bmatrix}$$

Since S is negative definite, system (5.9) is unstable.

<u>Example 5.2</u>: Consider the system

$$\left.\begin{aligned} z_1(\tau+1) &= a_1 z_1(\tau) + b_1\phi_1(\sigma_1(\tau)) \\ \sigma_1(\tau) &= c_1' z_2(\tau) \\ z_2(\tau+1) &= A_2 z_2(\tau) + b_2\phi_2(\sigma_2(\tau)) \\ \sigma_2(\tau) &= c_2 z_1(\tau) \end{aligned}\right\} \tag{5.10}$$

where $z_2 \varepsilon R^{n_2}$, A_2 is a $n_2 x n_2$ matrix, $z_1 \varepsilon R^1$, $a_1 \varepsilon R^1$, $|a_1|>1$, $||A_2|| < 1$, c_1, b_2 are n_2-vectors, c_2, b_1 are scalars and $\phi_i(\sigma_i)$, $i = 1,2$, are real, single valued functions such that $\phi_i(0) = 0$ and $0 < \frac{\phi_i(\sigma_i)}{\sigma_i} \leq k_i$, $\sigma_i \neq 0$, where k_1, k_2 are constants. System (5.10) is a discrete system analogous to the type of continuous systems considered in Chapter 4.

Let the first isolated subsystem be

$$z_1(\tau+1) = a_1 z_1(\tau)$$

with

$V_1 = -|z_1|$. Then

$$\Delta V_1 = |z_1| - |a_1 z_1|$$

and so

(i) $-|z_i| \leq V_1 \leq -|z_i|$

(ii) $\Delta V_1 \leq -(|a_1|-1)|z_1|$

(iii) $|V_1(\bar{z}_1,\tau) - V_1(z,\tau)| \leq |\bar{z}_1 - z_1|$.

Let the second isolated subsystem be

$$z_2(\tau+1) = A_2 z_2(\tau)$$

with

$V_2 = |z_2|$. Then

$$\Delta V_2 = |A_1 z_2| - |z_2|$$

and so

(i) $|z_2| \leq V_2 \leq |z_2|$

(ii) $\Delta V_2 \leq -(1-||A_2||)|z_2|$

(iii) $|V_2(\bar{z}_2,\tau) - V_2(z_2,\tau)| \leq |\bar{z}_2 - z_2|$

The interconnections may be bounded as

$$|g_{12}(z_2,\tau)| = |b_1\phi_1(c_1'z_2)| \le k_1|b_1||c_1||z_2| \quad \text{and}$$

$$|g_{21}(z_1,\tau)| = |b_2\phi_2(c_2'z_1)| \le k_2|b_2||c_2||z_1|.$$

Then choosing $\alpha_1 = \alpha_2 = 1$, one obtains

$$S = \begin{bmatrix} |a_1| - 1 & -k_1|b_1||c_1| \\ -k_2|b_2||c_2| & 1-||A_2|| \end{bmatrix}$$

For S to be positive definite, the following condition must hold

$$(|a_1|-1)(1-||A_2||) > k_1k_2|b_1||b_2||c_1||c_2|$$

The first isolated subsystem was unstable because of the assumption $|a_1| > 1$ and the second isolated subsystem was stable because $||A_2|| < 1$. However, the sufficient condition for the instability of the whole system requires that $|a_1|$ and $||A_2||$ be different from 1 by a factor dependent on the interconnections.

A Boundedness Theorem

Theorem 5.3: For the system (5.1) with decomposition (5.2), if, for the isolated subsystems $\dot{z}_i = f_i(z_i,t)$, $i\varepsilon M$, there exist Lyapunov functions $V_i(z_i,t)$ defined on $t\varepsilon J$ and z_i such that $|z_i| \ge B_i$ that satisfy the following conditions:

(i) $\phi_{i1}(|z_i|) \leq V_i(z_i,t) \leq \phi_{i2}(|z_i|)$

(ii) $DV_{i(5.3)} \leq -c_{i3}|z_i|^2$

(iii) $|\nabla V_i(z_i,t)| \leq c_{i4}|z_i|$

where ϕ_{i1}, $\phi_{i2}\varepsilon K$ and are radially unbounded and c_{i3}, c_{i4} are positive scalars; if $|g_{ij}(z_j,t)|\leq k_{ij}|z_j|$ for all z_j such that $|z_j|\geq B_j$, for all $t\varepsilon J$ and for all $i,j\varepsilon M$, $i\neq j$; and if there exists $\alpha_i>0$, $i\varepsilon M$, such that the matrix $S = (s_{ij})$ defined by

$$s_{ij} = \begin{cases} -\alpha_i c_{i3}, & \text{if} \quad i=j \\ \frac{1}{2}(\alpha_i c_{i4}k_{ij}+\alpha_j c_{j4}k_{ji}), & \text{if} \quad i\neq j \end{cases}$$

is negative definite; then the trajectories of the system (5.1) with decomposition (5.2) are uniformly bounded.

<u>Proof</u>: Let $V(x,t) = \sum_{i=1}^{m} \alpha_i V_i(z_i,t)$. Then

$$\sum_{i=1}^{m} \alpha_i \phi_{i1}(|z_i|) \leq V(x,t) \leq \sum_{i=1}^{m} \alpha_i \phi_{i2}(|z_i|)$$

for $t\varepsilon J$ and for all x such that $|z_i| \geq B_i$, $i\varepsilon M$. Since $\sum_{i=1}^{m} \alpha_i \phi_{ij}\varepsilon K$, $j = 1,2$, V is positive definite outside a region which includes the equilibrium point and is radially unbounded. Then

$$DV_{(5.1)} = \sum_{i=1}^{m} \alpha_i DV_{i(5.3)} + \sum_{i=1}^{m} \alpha_i \nabla V_i'(z_i,t) \sum_{\substack{j=1 \\ j\neq i}}^{m} g_{ij}(z_j,t)$$

$$\leq -\sum_{i=1}^{m} \alpha_i c_{i3} |z_i|^2 + \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j\neq 1}}^{m} \alpha_i c_{i4} k_{ij} |z_i| |z_j|$$

for $t\varepsilon J$ and z_i, z_j such that $|z_i| \geq B_i$, $|z_j| \geq B_j$. Let

$y' = [|z_1| \ldots |z_m|]$ and $R = (r_{ij})$ be an mxm matrix defined by

$$r_{ij} = \begin{cases} -\alpha_i c_{i3}, & \text{if } i=j \\ \alpha_i c_{i4} k_{ij}, & \text{if } i\neq j \end{cases}$$

Then

$$DV_{(5.1)} \leq y'Ry = \frac{1}{2} y'(R+R')y = y'Sy \leq \lambda_{max}(S)|x|^2$$

Hence $DV_{(5.1)}$ is negative for all $t\varepsilon J$ and x such that $|z_i| \geq B_i$, $i\varepsilon M$. Therefore, the solutions of (5.1) are uniformly bounded.

Analogous theorems for systems with linear and other kinds of interconnections can be formulated in a similar manner. Also, a similar theorem can be stated and proved for systems described by difference equations by making the usual modifications.

Application

Example 5.3: Consider the system (4.1) from Chapter 4 with the nonlinearities violating the sector conditions:

$$\left.\begin{aligned} \dot{z}_i &= A_i z_i - b_i \phi_i(\sigma_i) \\ y_i &= c_i' z_i \\ \sigma_i &= d_i' y = \sum_{j=1}^{m} d_{ij} y_j \end{aligned}\right\} \quad i = 1,\ldots,m \tag{5.11}$$

where A_i is an $n_i x n_i$ matrix, b_i and c_i are n_i-vectors, α_i is an m-vector and y_i and σ_i are scalars. Also, A_i is a stable matrix, (A_i,b_i) and (A_i',c_i) are completely controllable and $\phi_i(\sigma_i)$ is a real-valued piecewise continuous function of σ_i such that there exist real numbers $\bar{\sigma}_i \geq 0$, $k_i > 0$,

$$0 < \frac{\phi_i(\sigma_i)}{\sigma_i} < k_i, \qquad |\sigma_i| \geq \bar{\sigma}_i$$

and

$$\lim_{\sigma_i \to \infty} \int_0^{\sigma_i} \phi(\eta)d\eta \to \infty.$$

The ith isolated subsystem may be represented by

$$\begin{aligned} \dot{z}_i &= A_i z_i - b_i \phi_i(\sigma_i) \\ \sigma_i &= c_i' z_i \end{aligned} \tag{5.12}$$

and this is the direct control problem except that the nonlinear element satisfies the sector condition only for values of σ_i such that $|\sigma_i| \geq \bar{\sigma}_i$.

Wu and Manke [23] have shown that subsystem (5.12) is uniformly bounded if the Popov condition

$$\frac{\delta_i}{k_i} + \mathrm{Re}(\delta_i + j\omega\beta_i)G_i(j\omega) > 0$$

is satisfied. They show that the Lyapunov function constructed from the Popov line

$$V_i = z_i' P_i z_i + \beta_i \int_0^{\sigma_i} \phi_i(\eta) d\eta$$

is positive definite and radially unbounded and $DV_{i(5.12)}$ is negative definite for all $z_i \varepsilon R^{n_i}$ and $|\sigma_i| \geq m_i$, $m_i > 0$.

Let the Lyapunov function V for the composite system be $V = \sum_{i=1}^{m} \alpha_i V_i$. Then V is positive definite and radially unbounded and $DV_{(5.11)}$ is negative definite for all $x \varepsilon R^n$ and $|\sigma_i| > m_i$, $i = 1,\ldots,m$, if the matrix $\left[\begin{array}{c|c} \Omega & S \\ \hline S' & R \end{array}\right]$, defined in Chapter 4, is negative definite.

In Chapter 4 it is shown that system (4.1) is absolutely stable if each subsystem satisfies the Popov criterion and if the above partitioned matrix is negative definite. It is shown here that if the nonlinearities of this system are not always confined within the gain sector but the system satisfies the other conditions, the system is uniformly bounded.

CHAPTER 6. COMPENSATION

In this chapter a simple method for compensation of composite systems is outlined. In Chapter 5, theorems are stated and proved for the instability of composite systems when one or more of its isolated subsystems are unstable. In this chapter conditions for the stability of composite systems that have some unstable subsystems are established in a theorem. The nature of the S matrix in this theorem, that has to be negative definite for stability, suggests a method of compensation by adding local negative feedback to some of the subsystems. It must be pointed out here that this interconnected systems approach only establishes sufficient conditions for stability or instability. Sometimes it may not be possible to establish either stability or instability for a system by this method. Then, to guarantee stability it may be necessary to compensate the system even though it may be stable already, unless, of course, some other method of stability analysis can be found.

A Stability Theorem

Consider the system

$$\dot{x} = f(x,t) + g(x,t) = h(x,t), \quad f,g,h \in E \tag{6.1}$$

with decomposition

$$\dot{z}_i = f_i(z_i,t) + \sum_{j=1}^{m} g_{ij}(z_j,t), \quad i\varepsilon M = [1,\ldots,m] \tag{6.2}$$

The isolated subsystems can then be represented by

$$\dot{z}_i = f_i(z_i,t), \quad i\varepsilon M \tag{6.3}$$

Theorem 6.1: If for each isolated subsystem $\dot{z}_i = f_i(z_i,t)$, $i\varepsilon N\subset M$ there exists a Lyapunov function $V_i(z_i,t)$ with the following properties:

(i) $c_{i1}|z_i|^2 \leq V_i(z_i,t) \leq c_{i2}|z_i|^2$

(ii) $c_{i4}|z_i|^2 \leq DV_{i\,(6.3)} \leq c_{i3}|z_i|^2$

(iii) $\nabla V_i'(z_i,t)g_{ij}(z_j,t) \leq a_{ij}|z_i||z_j|, \quad j\varepsilon M$

for all $z_i\varepsilon R^{n_i}$, $z_j\varepsilon R^{n_j}$ and $t\varepsilon J$, where $c_{i1}, c_{i2}, c_{i3}, c_{i4}$ are positive scalars; if, for each isolated subsystem $\dot{z}_i = f_i(z_i,t)$, $i\not\in N$, $i\varepsilon M$, there exists a Lyapunov function $V_i(z_i,t)$ with the following properties:

(i) $c_{i1}|z_i|^2 \leq V_i(z_i,t) \leq c_{i2}|z_i|^2$

(ii) $DV_{i(6.3)}(z_i,t) \leq c_{i3}|z_i|^2$

(iii) $\nabla V_i'(z_i,t)g_{ij}(z_j,t) \leq a_{ij}|z_i||z_j|, \quad j\varepsilon M$

for all $z_i\varepsilon R^{n_i}$, $z_j\varepsilon R^{n_j}$ and $t\varepsilon J$, where c_{i1}, c_{i2} are positive scalars and c_{i3} is a negative scalar; and, if there exist $\alpha_i>0$, $i\varepsilon M$, such that the matrix $S = (s_{ij})$ defined by

$$s_{ij} = \begin{cases} \alpha_i(c_{i3}+a_{ii}), & \text{if } i=j \\ \frac{1}{2}(\alpha_i a_{ij}+\alpha_j a_{ji}), & \text{if } i\neq j \end{cases} \quad \text{is negative definite,}$$

then, the equilibrium $x=0$ of system (6.1) is uniformly exponentially stable in the whole.

<u>Proof</u>: Let $V(x,t) = \sum_{i=1}^{m} \alpha_i V_i(z_i,t)$

Then

$$\sum_{i=1}^{m} c_{i1}|z_i|^2 \leq V(x,t) \leq \sum_{i=1}^{m} c_{i2}|z_i|^2 \quad \text{for all} \quad x\varepsilon R^{\sum_{i=1}^{m} n_i}$$

and $t\varepsilon J$.

Now,

$$DV_{(6.1)} = \sum_{i=1}^{m} \alpha_i \frac{\partial V_i}{\partial t}(z_i,t) + \sum_{i=1}^{m} \alpha_i \nabla V_i(z_i,t)' f_i(z_i,t)$$

$$+ \sum_{i=1}^{m} \alpha_i \nabla V_i(z_i,t)' \sum_{j=1}^{m} g_{ij}(z_j,t)$$

$$= \sum_{i=1}^{m} \alpha_i DV_{i(6.3)} + \sum_{i=1}^{m} \alpha_i \nabla V_i(z_i,t)' \sum_{j=1}^{m} g_{ij}(z_j,t)$$

$$\leq \sum_{i=1}^{m} \alpha_i c_{i3}|z_i|^2 + \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i a_{ij}|z_i||z_j|$$

Choose

$$y' = [|z_1|\ldots|z_m|]$$

and matrix $R = (r_{ij})$ with

$$r_{ij} = \begin{cases} \alpha_i(c_{i3}+a_{ii}), & \text{if} \quad i=j \\ \alpha_i a_{ij}, & \text{if } i \neq j \end{cases}$$

Then

$$DV_{(6.1)} \leq y'Ry = \frac{1}{y} y'(R+R')y = y'Sy$$

Therefore,

$$DV_{(6.1)} \leq \lambda_{max}(S)|y|^2 = \lambda_{max}(S)|x|^2 \quad \text{where} \quad \lambda_{max}(S)$$

is negative because S is symmetric and negative definite. Therefore, the equilibrium x=0 of the system (6.1) is uniformly exponentially stable in the whole.

This theorem establishes the sufficient conditions for stability of the composite system that may (or may not) have some subsystems that are unstable. It should be noted that the isolated subsystems $\dot{z}_i = f_i(z_i,t)$, $i \varepsilon N \subset M$ are unstable and all the others are stable.

An analogous theorem for systems represented by the difference equations

$$x(\tau+1) = f[x(\tau),\tau] + g[x(\tau),\tau] = h[x(\tau),\tau]$$

with decomposition

$$z_i(\tau+1) = f_i[z_i(\tau),\tau] + \sum_{j=1}^{m} g_{ij}[z_j(\tau),\tau], \; i\varepsilon M$$

can be formulated in a similar manner by making the obvious modifications. The compensation scheme outlined in the next section for continuous systems can be applied similarly to discrete systems of this type.

Compensation

In Theorem 6.1 the final test of stability of system (6.1) is the negative definiteness of the matrix $S = (s_{ij})$. For S to be negative definite a necessary condition is that each diagonal element s_{ii} be negative. Now,

$$s_{ii} = \alpha_i(c_{i3}+a_{ii})$$

and α_i is positive. Hence, $s_{ii}<0$ implies

$$c_{i3} + a_{ii} < 0$$

The scalar c_{i3} is negative for stable isolated subsystems and positive for unstable isolated subsystems. The scalar a_{ii} is dependent on the feedback loop around the ith subsystem. Therefore, if there is no negative feedback around an unstable subsystem to make $s_{ii} < 0$, the composite system cannot be shown to be stable by the present method.

This suggests a method by which a system can be guaranteed to be stable. First, a well-known theorem [21] on the definitions of matrices with "dominant" main diagonal is stated:

For a real nxn matrix $B = (b_{ij})$, if

$$b_{ii} > \sum_{\substack{j=1 \\ j\neq 1}}^{n} |b_{ij}|, \quad i = 1,2,\ldots,n$$

then all the characteristic roots of B have positive real parts.

Since S is symmetric, its characteristic roots are all real. Hence a sufficient condition for S to be negative definite, that is, to have only negative characteristic roots, is that -S has a "dominant" main diagonal.

If S is not negative definite, the diagonal of -S can be made "dominant" by reducing the values of the n scalars, a_{ii}, $i = 1,\ldots,n$. This can be done by applying negative feedbacks $u_i(z_i,t)$ to the isolated subsystems. Since

$$\nabla V_i'(z_i,t) g_{ii}(z_i,t) \leq a_{ii}|z_i|^2 \qquad i = 1,\ldots,n$$

the addition of a negative feedback $u_i(z_i,t)$ such that

$$\nabla V_i'(z_i,t) u_i(z_i,t) \leq -k_i|z_i|^2, \qquad k_i>0$$

to the ith subsystem, reduces the value of a_{ii} by k_i.

A subsystem with interconnections is shown before and after compensation in Figures 6.1 and 6.2.

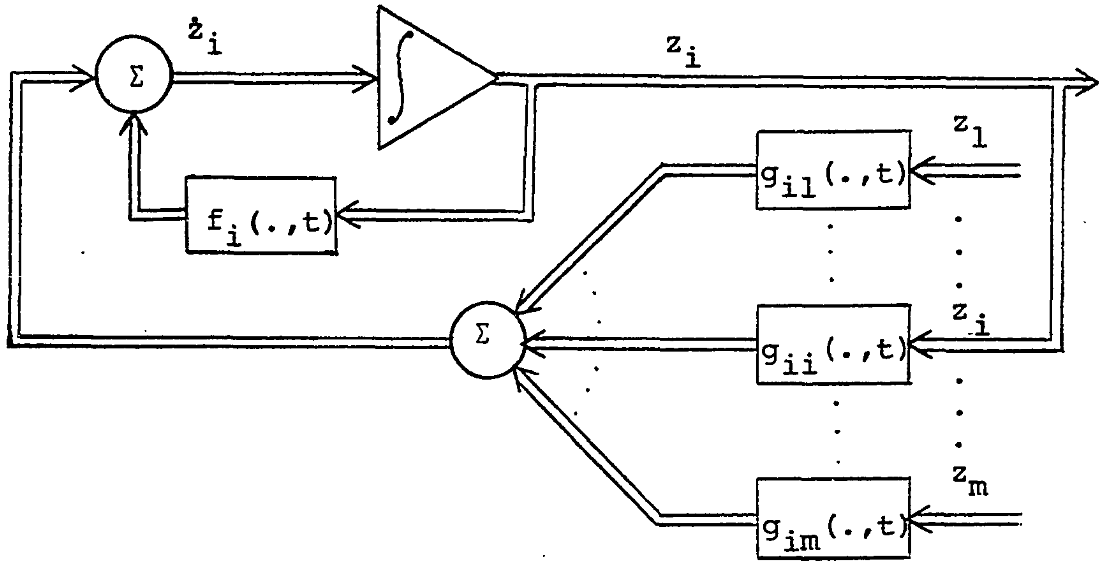

Figure 6.1. The ith subsystem of system (6.1) with interconnections

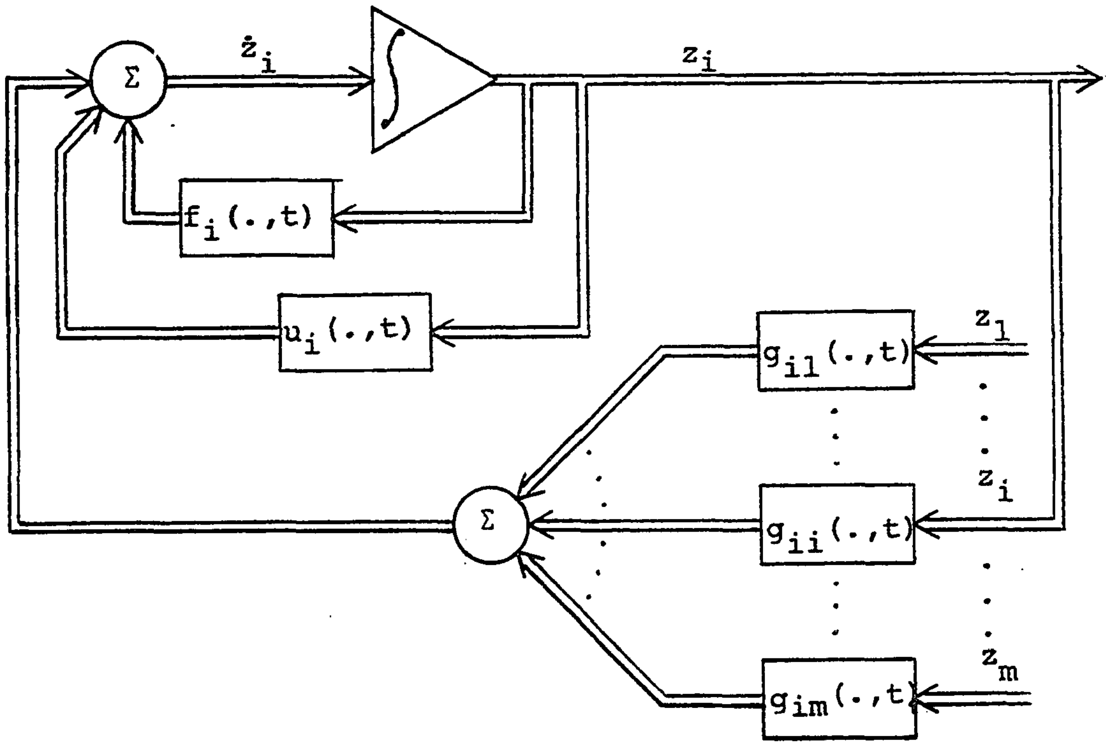

Figure 6.2. The ith subsystem of system (6.1) after compensation with interconnections

Applications

Example 6.1: Consider the system

$$\dot{z}_1 = f_1(z_1,t) + C_{12}z_2$$
$$\dot{z}_2 = f_2(z_2,t) + C_{21}z_1 \qquad (6.4)$$

Assume that for the first isolated subsystem

$$\dot{z}_1 = f_1(z_1,t)$$

there exists a Lyapunov function $V_1(z_1,t)$ such that

$$-c_{11}|z_1|^2 \leq V_1(z_1,t) \leq -c_{12}|z_1|^2$$

$$-\bar{c}_{13}|z_1|^2 \leq DV_1(z_1,t) \leq -c_{13}|z_1|^2$$

$$|\nabla V_1(z_1,t)| \leq c_{14}|z_1|$$

for all $z_1 \varepsilon R^{n_1}$ and $t\varepsilon J$ with c_{11}, c_{12}, c_{13}, $\bar{c}_{13}$, c_{14} positive scalars. This subsystem is obviously unstable. Also, assume that for the second isolated subsystem

$$\dot{z}_2 = f_2(z_2,t)$$

there exists a Lyapunov function $V_2(z_2,t)$ such that

$$c_{21}|z_2|^2 \leq V_2(z_2,t) \leq c_{22}|z_2|^2$$

$$DV_2(z_2,t) \leq -c_{23}|z_2|^2$$

$$|\nabla V_2(z_2,t)| \le c_{24}|z_2|$$

for all $z_2 \varepsilon R^{n_2}$ and $t \varepsilon J$ with c_{21}, c_{22}, c_{23}, c_{24} positive scalars. This subsystem is stable.

Then the hypotheses for Corollary 5.1 are satisfied and the matrix S has the elements

$$s_{11} = -\alpha_1 c_{13},\ s_{22} = -\alpha_2 c_{23},$$

$$s_{12} = s_{22} = \tfrac{1}{2}(\alpha_1 c_{14}||C_{12}||+\alpha_2 c_{24}||C_{21}||).$$

Choosing

$$\alpha_1 = \frac{1}{c_{14}||C_{12}||} \text{ and } \alpha_2 = \frac{1}{c_{24}||C_{21}||},$$ the matrix S becomes

$$S = \begin{bmatrix} -\dfrac{c_{13}}{c_{14}||C_{12}||} & 1 \\ 1 & -\dfrac{c_{23}}{c_{24}||C_{21}||} \end{bmatrix}$$

From Corollary 5.1, it follows that the system (6.4) is unstable provided that

$$||C_{12}||\ ||C_{21}|| < \frac{c_{13}c_{23}}{c_{14}c_{24}}$$

To compensate this system so that it will be stable a feedback $u_1(z_1,t)$ is needed for the first subsystem. For the first subsystem, choose a Lyapunov function

$$\overline{V}_1(z_1,t) = -V_1(z_1,t)$$

Then

$$c_{12}|z_1|^2 \le \overline{V}_1(z_1,t) \le c_{11}|z_1|^2$$

$$c_{13}|z_1|^2 \le D\overline{V}_1(z_1,t) \le \bar{c}_{13}|z_1|^2$$

$$\nabla\overline{V}_1'(z_1,t)g_{11}(z_1,t) = 0$$

$$\nabla\overline{V}_1'(z_1,t)g_{12}(z_2,t) \le c_{14}||C_{12}||\ |z_1||z_2|$$

For the second subsystem,

$$c_{21}|z_2|^2 \le V_2(z_2,t) \le c_{22}|z_2|^2$$

$$DV_2(z_2,t) \le -c_{23}|z_2|^2$$

$$\nabla V_2'(z_2,t)g_{21}(z_1,t) \le c_{24}||C_{21}|||z_2||z_1|$$

$$V_2'(z_2,t)g_{22}(z_2,t) = 0$$

This satisfies the hypothesis of Theorem 6.1. The S matrix of this theorem, on choosing the same value of α_1 and α_2 as before, becomes

$$S = \begin{bmatrix} \dfrac{\bar{c}_{13}}{c_{14}||C_{12}||} & 1 \\ 1 & -\dfrac{c_{23}}{c_{24}||C_{21}||} \end{bmatrix}$$

As would be expected for an unstable system this S is not negative definite. Now, apply the feedback $u_1(z_1,t)$ on the first subsystem such that

$$\nabla V_1'(z_1,t)u_1(z_1,t) \leq -k_1|z_1|^2 \tag{6.5}$$

Then, the new S matrix of Theorem 6.1 becomes

$$\begin{bmatrix} \frac{\bar{c}_{13}-k_1}{c_{14}||C_{12}||} & 1 \\ 1 & -\frac{c_{23}}{c_{24}||C_{21}||} \end{bmatrix}$$

which is negative definite if

$$k_1 > \bar{c}_{13} + ||C_{12}||\ ||C_{21}||\frac{c_{14}c_{24}}{c_{23}} \tag{6.6}$$

Hence the unstable system (6.4) after compensation becomes

$$\begin{aligned} \dot{z}_1 &= f_1(z_1,t) + u_1(z_1,t) + C_{12}z_2 \\ \dot{z}_2 &= f_2(z_2,t) + C_{21}z_1 \end{aligned} \tag{6.7}$$

which is stable if $u_1(z_1,t)$ satisfies the conditions (6.5) and (6.6).

<u>Example 6.2</u>: Consider the unstable system of Example (5.1) For the first subsystem choose

$$V_1 = -z_1'P_1z_1$$

where P_1 is the same as before. Then

$$0.38|z_1|^2 \leq V(z_1) \leq 2.62|z_1|^2$$

$$DV_1(z_1) \leq 6|z_1|^2$$

Let $V_2(z_2,t)$ and $V_3(z_3,t)$ be the same as before. Then

$$\nabla V_1'(z)\ g_{12}(z_2,t) = 0$$

$$\nabla V_1'(z_1)\ g_{13}(z_3,t) = 0$$

$$\nabla V_2'(z_2,t)\ g_{21}(z_1) \leq \frac{7}{4}\ |z_2|\ |z_1|$$

$$\nabla V_2'(z_2,t)\ g_{23}(z_3) \leq 7|z_2|\ |z_3|$$

$$\nabla V_3'(z_3,t)\ g_{31}(z_1,t) \leq 2|z_3|\ |z_1|$$

$$\nabla V_3'(z_3,t)\ g_{32}(z_2) \leq 3|z_3|\ |z_2|$$

and all $g_{ii}(z_i,t) = 0$, $i = 1,2,3$.

This satisfies the hypothesis of Theorem 6.1 and the S matrix is

$$S = \begin{bmatrix} 6 & \frac{7}{8} & 1 \\ \frac{7}{8} & -2 & 5 \\ 1 & 5 & -20 \end{bmatrix}$$

S is not negative definite as would be expected of an unstable system.

Choose a feedback $u_1(z_1)$ such that

$$\nabla V_1'(z_1)u_1(z_1) \leq -8|z_1|^2$$

for the first subsystem. Then the new S matrix is

$$\begin{bmatrix} -2 & \frac{7}{8} & 1 \\ \frac{7}{8} & -2 & 5 \\ 1 & 5 & -20 \end{bmatrix}$$

which is negative definite.

A $u_1(z_1)$ that will satisfy this condition is

$$u_1(z_1) = \begin{bmatrix} -4 & -4 \\ -4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

since

$$\nabla V_1(z_1) = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Hence, to render the system stable, the first two equations of the system have to be changed to

$$\dot{x}_1 = 2x_1 + 2x_2 - 4x_1 - 4x_2 = -2x_1 - 2x_2$$

$$\dot{x}_2 = x_1 + 3x_2 - 4x_1 - 8x_2 = -3x_1 - 4x_2$$

CHAPTER 7. CONCLUSIONS

Results are obtained for stability, instability and boundedness for systems which may be considered to be composed of subsystems of lower orders appropriately interconnected. A particular result is used to develop a simple method of compensation for unstable systems of this type. The Lyapunov function for such a system is taken to be the weighted sum of the Lyapunov functions of its subsystems. The qualitative behavior of the system is then determined from the interconnections and the Lyapunov functions of the subsystems.

A problem in using this method is the determination of appropriate Lyapunov functions for the subsystems. It is shown that for a system which is made up of subsystems described by the regulator equations, the Popov criterion can be used. If this frequency domain criterion is used to determine the stability of each subsystem, the Kalman-Yakubovich Lemma provides a method of constructing a Lyapunov function of the Luré type for each subsystem. The Lyapunov function for the composite system is then determined and the conditions for stability of such a system are established. Since the stability of this class of systems can also be analyzed by using the multidimensional Popov criterion, the two methods are compared. It is shown that the main condition for stability in using the method developed here is the determination of a negative definite symmetric matrix of scalars whereas in using the multidimensional Popov criterion, which does not have a simple geometric construction, a positive matrix of transfer functions has to be determined.

Conditions are established for instability of interconnected composite systems. It is shown that the composite system is unstable if one or more of its subsystems are unstable together with some other conditions. Also, conditions are established for the trajectories of a composite system to be bounded. It is shown that if the above system that is composed of regulator-type subsystems have nonlinearities that are not always confined to certain gain sectors, the system is bounded but not absolutely stable.

For unstable composite systems a simple method of compensation is developed. First, conditions are established for stability of a system when some of its subsystems are unstable and the others are stable. An unstable system cannot satisfy these conditions but it is shown that by adding negative feedback to some of the subsystems the conditions can be met. A scheme to determine the gain of the compensating feedback loop is developed.

Examples are included to illustrate these results.

In the course of this work certain questions have arisen which remain unresolved. Some of these are as follows: (i)

Since the Direct Method of Lyapunov, in general, provides only sufficient conditions for the qualitative behavior of a system, some conservativeness is expected in the answers. Given that such conservativeness exists in the choice of Lyapunov functions for the subsystems, do the results obtained here, which are also sufficient, introduce another degree of conservatism in the answers for the composite system? If it does, can it be measured?

(ii) If a system is decomposed such that no subsystem has interconnections with itself, the results obtained indicate that the composite system is stable when all the subsystems are stable and some other conditions are satisfied. Such a system is unstable when at least one subsystem is unstable and similar conditions are satisfied. This leaves a large gray area where these extra conditions are not met and where the system cannot be shown to be either stable or unstable. Can this be improved?

(iii) Because of this conservatism the compensation scheme may require feedback gains much higher than what is necessary. Also, only negative feedback for the subsystems was tried for compensation. Can better schemes for compensation be devised? Some method of modification of the interconnecting structure to attain stability may hold some promise.

(iv) The results obtained are dependent on the judicious choice of the weighting factors for the summation of the Lyapunov functions for the subsystems. Can a method be developed for the choice of the optimal weighting factors? Since one usual condition for the desired behavior of the system is the definiteness of a constructed matrix which is dependent on these weighting factors, the optimal choice has to be linked with the definiteness of the matrix.

(v) For the class of systems that utilizes the Popov criterion to find the Lyapunov functions for their subsystems, stability is conditioned on the negative definiteness of the matrix $\left[\begin{array}{c|c} Q & S \\ \hline S' & R \end{array}\right]$. However it is already known that partitions of this matrix, Q_i, $i = 1,\ldots,m$, and

$$\left[\begin{array}{c|c} Q_i & s_{ii} \\ \hline s'_{ii} & r_{ii} \end{array}\right], \quad i = 1,\ldots,m,$$ are negative definite since each subsystem is stable. Can this knowledge be utilized to find conditions on the other elements of S and R that will guarantee negative definiteness of the whole matrix?

REFERENCES

1. M. A. Aizerman and F. R. Gantmacher. Absolute Stability of Regulator Systems. San Francisco: Holden-Day, 1964.

2. B. D. O. Anderson. "Stability of Control Systems with Multiple Nonlinearities." Journal of the Franklin Inst., 282, No. 3 (September 1966):155-160.

3. M. Araki, K. Ando and B. Kondo. "Stability of Sampled Data Systems with Many Nonlinearities." IEEE Transactions on Automatic Control, AC-16, No. 1 (February 1971):22-27.

4. M. Araki and B. Kondo. "Stability and Transient Behavior of Composite Nonlinear Systems." IEEE Transactions on Automatic Control, AC-17, No. 4 (August 1972):537-541.

5. F. N. Bailey. "The Application of Lyapunov's Second Method to Interconnected Systems." J. SIAM Control, Ser. A, 3, No. 3 (1965):443-462.

6. Lj. T. Grujić and D. D. Siljak. "On Stability of Discrete Composite Systems." IEEE Transactions on Automatic Control, AC-18, No. 5 (October 1973):522-524.

7. Lj. T. Grujić and D. D. Siljak. "Asymptotic Stability and Instability of Large-Scale Systems." IEEE Transactions on Automatic Control, AC-18, No. 6 (December 1973):636-645.

8. W. Hahn. Stability of Motion. New York: Springer-Verlag, 1967.

9. E. S. Ibrahim and Z. V. Rekasius, "A Stability Criterion for Nonlinear Feedback Systems." IEEE Transactions on Automatic Control, AC-9, No. 2 (April 1964):154-159.

10. E. I. Jury and B. W. Lee. "The Absolute Stability of Systems with Many Nonlinearities." Automation and Remote Control, 26, No. 6 (June 1965):943-961.

11. R. E. Kalman. "Lyapunov Functions for the Problem of Luré in Automatic Control." Proc. National Academy of Sciences, 49, No. 2 (February 1963):201-205.

12. S. Lefschetz, Stability of Nonlinear Control Systems. New York: Academic Press, 1965.

13. A. N. Michel and D. W. Porter. "Stability Analysis of Composite Systems." IEEE Transactions on Automatic Control, AC-17, No. 2 (April 1972):222-226.

14. A. N. Michel. "Stability Analysis of Interconnected Systems," to appear in J. SIAM Control, Ser. A, 12, No. 5 (August 1974).

15. A. N. Michel. "Stability Analysis and Trajectory Behavior of Composite Systems." IEEE International Symp. on Circuits and Systems, pp. 240-244, 1974. To appear in IEEE Transactions on Circuit Theory, (April 1975).

16. A. N. Michel. "Stability Analysis of Stochastic Large-scale Systems." Eighth Annual Princeton Conference on Information Sciences and Systems, 1974. To appear in IEEE Transactions on Automatic Control.

17. K. S. Narendra and R. M. Goldwyn. "A Geometrical Criterion for the Stability of Certain Nonlinear Non-autonomous Systems." IEEE Transactions on Circuit Theory, CT-11, No. 9 (September 1964):406-408.

18. K. S. Narendra and C. P. Neuman. "Stability of Continuous Time Dynamical Systems with m-Feedback Nonlinearities." JACC, pp. 823-831, 1966.

19. A. A. Piontkovskii and L. D. Rutkovskaya. "Investigation of Certain Stability Theory by the Vector Lyapunov Function Method." Automation and Remote Control 28, No. 10 (October 1967)1422-1429.

20. D. W. Porter and A. N. Michel. "Bounded Input and Bounded Output Stability of Nonlinear Time-varying Multiloop Systems." JACC, 1974. Also, IEEE Transactions on Automatic Control, AC-19, No. 4 (August 1974): 422-427.

21. O. Taussky. "A Recurring Theorem on Determinants." American Math Monthly, 56, No. 12 (December 1949): 672-676.

22. W. E. Thompson. "Exponential Stability of Interconnected Systems." IEEE Transactions on Automatic Control, AC-15, No. 4 (August 1970):504-506.

23. S. H. Wu and G. M. Manke. "Popov Criterion and Lagrange Stability of Nonlinear Systems." IEEE Transactions on Automatic Control, AC-12, No. 5 (October 1967):627.

24. V. A. Yakubovich. "The Solution of Certain Matrix Inequalities in Automatic Control Theory." Dokl. Akad. Nauk USSR, 143, (1962):1304-1307.

25. T. Yoshizawa. Stability Theory by Lyapunov's Second Method. Tokyo: The Mathematical Society of Japan, 1966.

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APPENDIX: THE KALMAN-YAKUBOVICH LEMMA

In this appendix, the proof of Theorem 3.9 is presented. This theorem shows the equivalence of the Popov's criterion and Luré's Lyapunov function for the regulator system with one nonlinearity. The most important step in the proof is the Kalman-Yakubovich Lemma which is also stated and proved here. A method of constructing the Luré Lyapunov function from the Popov condition is developed in the lemma and the steps are clearly outlined separately at the end of this appendix. Although this material is available in the more recent literature [11], [12], [24], the presentation here is somewhat different in approach and matches the rest of this dissertation more closely.

Theorem 3.9: For the system represented by

$$\begin{aligned} \dot{x} &= Ax - b\phi(\sigma) \\ \sigma &= c'x \end{aligned} \tag{A.1}$$

where A is an nxn stable matrix, b and c are n-vectors, (A,b) and (A',c) are completely controllable and $\phi: R^1 \to R^1$, $\phi(\sigma)$ is piecewise continuous, $\phi(0) = 0$,

$$0 < \frac{\phi(\sigma)}{\sigma} \leq k, \ \sigma \neq 0, \ 0<k\leq\infty,$$

the (Luré) Lyapunov function,

$$V = x'Px + \beta \int_0^\sigma \phi(\eta)d\eta \tag{A.2}$$

where P is an nxn symmetric, positive matrix and β is a scalar, is positive definite and radially unbounded and its derivative,

$$DV_{(A.1)} = -x'Qx - 2\phi(Pb-\ell)'x - \tau\phi^2 - \delta(\sigma - \frac{\phi}{k})\phi \tag{A.3}$$

where

$Q = -(A'P+PA)$ is an nxn matrix (necessarily symmetric and positive definite),

$\ell = \frac{1}{2}\beta A'c + \frac{1}{2}\delta c$ is an n-vector,

$\tau = \beta c'b + \frac{\delta}{k}$ and δ are scalars,

is negative definite if and only if

$$\frac{\delta}{k} + \mathrm{Re}\{(\delta + j\omega\beta)c'(j\omega I - A)^{-1}b\} > 0 \tag{A.4}$$

for all $\omega \geq 0$ together with

(i) $\delta>0$, $\beta>0$

and

(ii) $\tau>0$ or $\tau=0$, $Pb-\ell = 0$.

Before the proof of this theorem is presented the following lemma is stated and proved:

Kalman-Yakubovich Lemma: Let A be a matrix whose eigenvalues have negative real parts, D be a symmetric, positive definite matrix, b and ℓ be vectors such that (A,b) and (A',ℓ) are completely controllable, and $\tau \geq 0$ and $\varepsilon > 0$ be scalars. Let the vector q and matrix P (necessarily symmetric and positive definite) be connected by the equations:

$$A'P + PA = -qq' - \varepsilon D \tag{A.5}$$

$$Pb - \ell = \sqrt{\tau} q \tag{A.6}$$

Then a pair (q,P) exists if and only if

$$\tau + 2\mathrm{Re}\{\ell'(j\omega I - A)^{-1} b\} > 0 \tag{A.7}$$

is satisfied for all ω, $0 \leq \omega < \infty$.

Proof: The necessity of Condition (A.7):

Suppose

P,q satisfy Conditions (A.5) and (A.6). Then (A.5) yields

$$j\omega P - PA - j\omega P - A'P = qq' + \varepsilon D. \tag{A.8}$$

Let

$S = j\omega I - A$. Then $S^* = -j\omega I - A'$

and (A.8) may be written

$$PS + S^*P = qq' + \varepsilon D \tag{A.9}$$

Premultiplying by $b'S^{*-1}$ and postmultiplying by $S^{-1}b$ and substituting Pb and b'P from (A.6), (A.9) may be written as

$$b'S^{*-1}(\ell+\sqrt{\tau}q) + (\ell+\sqrt{\tau}q)'S^{-1}b$$

$$= b'S^{*-1}qq'S^{-1}b + \varepsilon b'S^{*-1}DS^{-1}b$$

Since each term is now a scalar, the terms may be transposed:

$$\ell'\overline{S}^{-1}b + \ell'S^{-1}b + \sqrt{\tau}q'\overline{S}^{-1}b + \sqrt{\tau}q'S^{-1}b$$

$$= (q'\overline{S}^{-1}b)(q'S^{-1}b)' + \varepsilon b'S^{*-1}DS^{-1}b$$

i.e. $$2\mathrm{Re}\{\ell'S^{-1}b\} + 2\sqrt{\tau}\mathrm{Re}\{q'S^{-1}b\} = |q'S^{-1}b|^2 + \varepsilon b'S^{*-1}DS^{-1}b$$

i.e. $$\tau+2\mathrm{Re}\{\ell'S^{-1}b\} = |q'S^{-1}b-\sqrt{\tau}|^2 + \varepsilon b'S^{*-1}DS^{-1}b \qquad (A.10)$$

Since D is Hermitian and positive definite, $S^{*-1}DS^{-1}$ is also Hermitian and positive definite. Hence, the right hand side of (A.10) is positive and so

$$\tau+2\mathrm{Re}\{\ell'(j\omega I-A)^{-1}b\} > 0 \quad \text{for} \quad 0\leq\omega<\infty$$

The sufficiency of Condition (A.7):

Suppose (A.7) holds. Then let

$$\nu = \text{lower bound } [2\mathrm{Re}\{\ell'(i\omega I-A)^{-1}b\}]$$

and

$$\mu = \text{upper bound } [b'S^{*-1}DS^{-1}b].$$

Since $\mathrm{Re}\{\ell'S^{-1}b\}$ and $b'S^{*-1}DS^{-1}b$ are both real rational continuous functions of ω with the numerator having a lower degree than the denominator, both ν and μ exist.

Then choosing $\varepsilon < (\tau+\nu)/\mu$, (A.7) may be changed to

$$\tau + 2\mathrm{Re}\{\ell'S^{-1}b\} - \varepsilon b'S^{*-1}DS^{-1}b > 0 \qquad \text{(A.11)}$$

Then define $\psi(s) = |sI-A| = s^n + a_n s^{n-1} + \ldots + a_1$

and

$$e_n = b$$

$$e_{n-1} = (A+a_nI)b$$

$$\vdots$$

$$e_1 = (A^{n-1}+a_nA^{n-2} + \ldots + a_2I)b$$

Since (A,b) is completely controllable, $b, Ab, \ldots, A^{n-1}b$ are linearly independent. Therefore, $e_1, e_2, \ldots, e_n$ are linearly independent and form a basis on which A and b may be written as

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -a_1 & -a_2 & -a_3 & \ldots & -a_n \end{bmatrix} \text{ and } \tilde{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

that is, $\tilde{A} = L^{-1}AL$ and $\tilde{b} = L^{-1}b$ (A.12) where

$$L = [e_1 \;\; e_2 \; \cdots \; e_n].$$

To show (A.12), $L\tilde{A} = [e_1 \;\; e_2 \; \cdots \; e_n]\tilde{A}$

$$= [-a_1 e_n \;\; e_1 - a_2 e_n \; \cdots \; e_{n-1} - a_n e_n]$$

$$= [Ae_1 \;\; Ae_2 \; \cdots \; Ae_n]$$

$$= AL$$

and

$$L\tilde{b} = [e_1 \;\; e_2 \; \cdots \; e_n]\tilde{b} = e_n = b.$$

Let

$\tilde{h} = L'h = [\tilde{h}_1 \ldots \tilde{h}_n]$ for any $h \varepsilon R^n$ and then

$$h'(sI-A)^{-1}b = h'LL^{-1}(sI-A)^{-1}LL^{-1}b = \tilde{h}'(sI-\tilde{A})^{-1}\tilde{b}$$

$$= [\tilde{h}_1 \ldots \tilde{h}_n] \begin{bmatrix} s & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & (a_n+s) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$= \frac{\tilde{h}_1 + \tilde{h}_2 s + \ldots + \tilde{h}_n s^{n-1}}{\psi(s)} \tag{A.13}$$

Since the inequality (A.11) has a real rational function in ω on the left hand side, it may be written as

$$\tau + 2\mathrm{Re}\{\ell' S^{-1} b\} - \varepsilon b' S^{*-1} D S^{-1} b = \left|\frac{\eta(j\omega)}{\psi(j\omega)}\right|^2 \tag{A.14}$$

Substituting

$$S^{-1} = (j\omega I-A)^{-1} = -A(\omega^2 I+A^2)^{-1}-j\omega(\omega^2 I+A^2)^{-1}$$

and

$$|\psi(j\omega)|^2 = \chi(j\omega)\chi(-j\omega) = |j\omega I-A||-j\omega I-A| = |\omega^2 I+A^2|$$

(A.14) may be written

$$[\tau-2\ell' A(\omega^2 I+A^2)^{-1}b-\varepsilon b' S^{*-1}DS^{-1}b]|\omega^2 I+A^2| = |\eta(j\omega)|^2 \tag{A.15}$$

Since the left hand side of (A.15) is a polynomial in ω^2, it may be decomposed into factors of the form

$$\omega^2 + \alpha^2 = |j\omega+\alpha|^2$$

and

$$\omega^4+2\omega^2(\rho^2-\gamma^2)+(\rho^2+\gamma^2)^2 = |(j\omega)^2+2j\omega\rho+(\rho^2+\gamma^2)|^2$$

and Equation (A.15) may be written as

$$|\eta(j\omega)|^2 = \tau\Pi[\omega^2+\alpha^2]\Pi[\omega^4+2\omega^2(\rho^2-\gamma^2)+(\rho^2+\gamma^2)^2]$$

Then,

$$|\eta(j\omega)| = \sqrt{\tau}\Pi[j\omega+\alpha]\Pi[(j\omega)^2+2j\omega\rho+(\rho^2+\gamma^2)]$$

Let $\tilde{q}$ be the vector whose elements are the coefficients of the (n-1) degree polynomial $\sqrt{\tau}\psi(j\omega)-\eta(j\omega)$. Then by (A.13)

$$\tilde{q}'(j\omega I-A)^{-1}\tilde{b} = \frac{\sqrt{\tau}\psi(j\omega)-\eta(j\omega)}{\psi(j\omega)} = q'(j\omega I-A)^{-1}b$$

where $q = L'^{-1}\tilde{q}$. Therefore,

$$|q'(j\omega I-A)^{-1}b-\sqrt{\tau}|^2 = |\frac{\eta(j\omega)}{\psi(j\omega)}|^2$$

$$= \tau+2\mathrm{Re}\{\ell'S^{-1}b\} -\varepsilon b'S^{*-1}DS^{-1}b$$

that is,

$$\tau+2\mathrm{Re}\{\ell'S^{-1}b\} = |q'S^{-1}b-\sqrt{\tau}|^2 + \varepsilon b'S^{*-1}DS^{-1}b$$

This is Equation (A.10) and so by retracing the steps of the necessity proof backwards, it may be shown that the constructed q satisfies (A.5) and (A.6). However, q is not unique but convenient. This completes the proof of the lemma.

Before proving Theorem 3.9 it has to be shown that Equations (A.4) and (A.7) are equivalent:

Substituting for τ and Ω, (A.7) may be written

$$\beta c'b + \frac{\delta}{k} + 2\mathrm{Re}\{(\frac{1}{2}\ \beta A'c + \frac{1}{2}\delta c)'(j\omega I-A)^{-1}b\} > 0,$$

that is,

$$\frac{\delta}{k} + \beta c'b + \mathrm{Re}\{\beta c'A(j\omega I-A)^{-1}b + \delta c'(j\omega I-A)^{-1}b\} > 0,$$

that is,

$$\frac{\delta}{k} + \beta c'b + \mathrm{Re}\{j\omega\beta c'(j\omega I-A)^{-1}b-\beta c'b+\delta c'(j\omega I-A)^{-1}b\} > 0$$

which on rearranging, becomes (A.4)

$$\frac{\delta}{k} + \mathrm{Re}\{(\delta+j\omega\beta)c'(j\omega I-A)^{-1}b\} > 0$$

Proof of Theorem 3.9: The necessity of Condition (A.4): Suppose V of (A.2) is positive definite and radially unbounded and $DV_{(A.1)}$ of (A.3) is negative definite. Then Condition (i) must hold. If $\tau>0$ and $Pb-\ell = \sqrt{\tau}q$, then

$$DV_{(A.1)} = -x'(Q-qq')x - (q'x + \sqrt{\tau}\phi)^2 - \delta(\sigma - \frac{\phi}{k})\phi \qquad (A.16)$$

which implies that $Q-qq' = \varepsilon D$ is positive definite. Hence (A.5) and (A.6) hold and then by the Lemma (A.7), that is, (A.4) must hold. If $\tau=0$ and $Pb-\ell = 0$, then

$$DV_{(A.1)} = -x'Qx-\delta(\sigma - \frac{\phi}{k})\phi \qquad (A.17)$$

which implies that $Q= qq' + \varepsilon D$ is positive definite. Again by the Lemma (A.7) and hence (A.4) are satisfied.

The sufficiency of Condition (A.4): Suppose (A.4) holds together with conditions (i) and (ii). Then (A.7) holds and by the lemma a pair (q,P) exists satisfying (A.5) and (A.6). If $\tau>0$, $DV_{(A.1)}$ may be written in (A.16). Since $D = Q-qq'$ is positive definite, $DV_{(A.1)}$ is negative definite. Since $Q = qq' + \varepsilon D$ is positive, P is postive definite, which together with condition (i) implies that V is positive definite. If $\tau=0$, $Pb-\ell = 0$, then $DV_{(A.1)}$ may be written as in (A.17). Since Q is positive definite, P must be positive definite and together with condition (i), this implies that $DV_{(A.1)}$ is negative definite and V is positive definite and radially unbounded. This completes the proof of the theorem.

The method of constructing the Luré Lyapunov function (A.2) for system (A.1) from the Popov criterion (A.4) is developed in the sufficiency proof of the Kalman-Yakubovich Lemma. For convenience it is presented here in a step by step method:

(1) Given the data for system (A.1), that is, A, b, c and k

(2) From the Popov criterion (A.4), δ and β are obtained, δ/β being the slope of the Popov line

(3) ℓ, τ, ν, μ and ε are calculated as follows

$$\ell = \frac{1}{2}\beta A'c + \frac{1}{2}\delta c$$
$$\tau = \beta c'b + \frac{\delta}{k}$$
$$\nu = \text{lower bound } [2\mathrm{Re}\{\ell'(j\omega I-A)^{-1}b\}]$$
$$\mu = \text{upper bound } [b'(-j\omega I-A)^{-1}D(j\omega I-A)^{-1}b]$$

where D is an arbitrary, symmetric, positive definite matrix (a frequent choice of D is the identity matrix),

$$\varepsilon < \frac{\tau+\nu}{\mu}$$

(4) The polynomial in ω^2

$$\frac{1}{\tau}[\tau-2\ell'A(\omega^2I+A^2)^{-1}b-\varepsilon b'(-j\omega I-A')^{-1}D(j\omega I-A)^{-1}b]\,|\omega^2I+A^2|$$

is decomposed into the forms $\omega^2+\alpha^2$ and $\omega^4+2(\rho^2-\gamma^2)\omega^2+(\rho^2+\gamma^2)$

(5) The n-vector $\tilde{q}$ is made up of the coefficients of

the (n-1) polynomianl $\sqrt{\tau}\psi(s)-\eta(s)$, where

$$\psi(s) = |sI-A|$$

and

$$\eta(s) = \sqrt{\tau}\Pi(s+\alpha)\Pi(s^2+2\rho s+\rho^2+\gamma^2)$$

(6) $q = M^{-1}\tilde{q}$ is calculated where

$$M = \begin{bmatrix} b'(A^{n-1}+a_nA^{n-2} +\ldots+ a_2I)' \\ \vdots \\ b'(A+a_nI)' \\ b' \end{bmatrix}$$

(7) P is then calculated from $A'P+PA = -qq'-\varepsilon D$

(8) P may be checked by $Pb-\ell = \sqrt{\tau}q$